June, 2021: Eigenvalues, Eigenvectors
SWMS-Worksheet-III in Linear Algebra

---

(1) Let W be a subspace of $\mathbb{R}^4$ given by

$$W = \{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : x - 2y + z + w = 0\}.$$

Find a basis of W. What is the dimension of W? Can you realize W as the null space of a linear transformation?

(2) Consider the map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $T(X) = AX$, where

$$A = \begin{bmatrix} 0 & e & \pi \\ \sqrt{2} & 0 & 1 \\ 0 & -e & \pi \end{bmatrix}.$$

(a) Is the map surjective? What does this say about solutions to $AX = B$ for arbitrary $B \in \mathbb{R}^3$?
(b) Is the map injective? What does this say about the uniqueness of the solutions to $AX = B$?

(3) Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -1 & 1 & 0 \\ 3 & 6 & -1 & 4 & 1 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}.$$

(4) Define $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ by

$$T(p)(x) = xp'(x)$$

for all x. Find all eigenvalues and eigenvectors of T.

(5) Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & -3 \\ -3 & 6 & 5 \end{bmatrix}.$$

Find the eigenspace of each eigenvalue.

(6) Let A and B be $n \times n$ matrices. Show that the $\text{rank}(AB) \leq \text{rank}(A)$

(7) Fins a 2×2 matrix A that has an eigenvalue $\lambda_1 = 1$ with eigenvector $v_1 = (1\ 2)^t$ and an eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = (2\ 1)^t$.