SWMS 2021

Discrete Geometry Worksheet II

In this worksheet, we will learn to recognize linear spans, affine hulls and convex hulls in 2 and 3-dimensional spaces.

Can you identify each of the following spaces as either

- $\text{lin}(v_1, ..., v_k)$, or
- $\text{aff}(v_1, ..., v_k)$, or
- $\text{cvx}(v_1, ..., v_k)$,

for some choice of vectors $v_1, ..., v_k$? You must specify the choice of vectors, which may vary from case to case. **Warning!** There may be examples that cannot be expressed as any of the three choices above.

(a) The null space of

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}.$$

(b) The solution space of the system of equations

$$x + 2z = 1$$
$$3x + y + z = 2.$$

A. Based on Parts (a) and (b), do you want to make a general statement about null spaces and solutions spaces of systems of linear equations?

(c) $\{(x, y) \in \mathbb{R}^2 : x - y \leq 0,\ y \geq 0,\ x + 2y \leq 1\ \}$.

(d) $\{(x, y) \in \mathbb{R}^2 : x - y \leq 0,\ y \geq 0,\ x + 2y \geq 1\ \}$.

(e) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

For the next three parts, $e_0 = (0, 0)$, $e_1 = (1, 0)$, and $e_2 = (0, 1)$.

(f) $\text{cvx}(e_0, e_1, e_2) \cap \text{cvx}(e_0, e_1 + e_2, e_2)$.

Definition. Given two sets $A, B \subset \mathbb{R}^n$, their *Minkowski sum* is the set

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

(g) $\text{cvx}(e_0, e_1, e_2) + \text{cvx}(e_0, e_1 + e_2, e_2)$.

B. Based on Part (f), do you have a conjecture for

$$\text{cvx}(v_1, ..., v_k) + \text{cvx}(w_1, ..., w_m)?$$

(h) $\text{cvx}(e_0, e_1, e_2) \cup \text{cvx}(e_0, e_1 + e_2, e_2)$.