SWMS 2021

Discrete Geometry Worksheet I

In this worksheet, we will explore linear, affine and convex combinations in the 2-dimensional plane.

(a) Let $v_1 = (1, 0)$ and $v_2 = (0, 1)$ in $\mathbb{R}^2$. Describe (with pictures and in set notation)

$$\text{lin}(v_1, v_2) \qquad \text{aff}(v_1, v_2) \qquad \text{cvx}(v_1, v_2). \tag{1}$$

(b) What is the effect on the three sets when you add the vector $v_3 = (1, 1)$ to the collection? What if $v_3 = (2, -1)$ instead?

(c) Given a set $A \subset \mathbb{R}^2$ and vector $b \in \mathbb{R}^2$,

$$A + b = \{a + b : a \in A\}.$$

Show that $\text{aff}(v_1, v_2) = \text{lin}(v_1 - v_2) + v_2$ for any pair of vectors $v_1, v_2 \in \mathbb{R}^2$.

A. I claim that $\text{aff}(v_1, ..., v_k) = \text{lin}(w_1, ..., w_{k-1}) + w_0$. Can you conjecture what $w_0, w_1, ..., w_{k-1}$ are in terms of $v_1, ..., v_k$?

(d) Let $C = \text{cvx}(v_1, v_2, v_3)$, where $v_1 = (1, 0)$, $v_2 = (0, 1)$, & $v_3 = (0, 0)$. Describe the sets

$$\text{cvx}(C, 2) = \bigcup_{w_1, w_2 \in C} \text{cvx}(w_1, w_2),$$

$$\text{cvx}(C, 3) = \bigcup_{w_1, w_2, w_3 \in C} \text{cvx}(w_1, w_2, w_3).$$

B. Define $\text{cvx}(C, n)$ for any positive integer n and formulate a conjecture about it.

(e) Observe that

$$\text{cvx}(v_1, ..., v_k) \subset \text{aff}(v_1, ..., v_k) \subset \text{lin}(v_1, ..., v_k).$$

Is it ever possible that all three sets are equal? How about the first two? How about the last two?

(f) What shape do you expect the convex hull of k vectors in $\mathbb{R}^2$ to be?