LECTURE - 3

Determinant of a square matrix A:

Let A be a $n \times n$ matrix, and let A_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained from A by crossing out the ith row and jth column of A:

Determinant of A is defined recursively by

- $\det[a] = 1$ where $[a]$ is a 1×1 matrix, and
- $\det A = a_{11} \det A_{11} - a_{21} \det A_{21} + a_{31} \det A_{31} - \cdots + (-1)^{n+1} \det A_{n1}$

(expansion by minors on the first column)

In fact, if we expand by minors on any column (or any row) we get the same value $\det A$. This is an important result, which I shall not prove here.

Exercise: Find the determinants of the elementary matrices.

Solution: Verify that if E is of type I, $\det E = 1$; if E is of type II, $\det E = -1$; and if E is of type III, where c is nonzero scalar replacing a diagonal entry of identity matrix, then $\det E = c$. Note that, in any case, $\det E$ is non-zero.

Determinant is multiplicative. I shall not prove this for you, but state it as a theorem.

Theorem: Let A, B be $n \times n$ matrices. Then $\det(AB) = \det A \cdot \det B$.

Corollary

ⓐ A square matrix is invertible if and only if its determinant is non-zero. If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.

ⓑ $\det A = \det(A^t)$, where A^t is the transpose of A.

Proof. If A is invertible, then A is the product of elementary matrices, say $A = E_1 \cdots E_k$.

Then, $\det A = (\det E_1) \cdots (\det E_k) \neq 0$ as $\det E_i \neq 0, \forall i$.

If A is not invertible, then the row ~~de~~ reduced echelon form A' has last row zero. So, $\det A' = 0$ (by expanding along last row).

But, there exist elementary matrices $E_1, \ldots, E_r$ such that $A' = E_1 \cdots E_r A$.

By multiplicative property of determinants,

$0 = \det A' = \det(E_1) \cdots \det(E_r) \det A$.

But $\det E_i \neq 0, \ \forall i = 1, \ldots, r$.

$\Rightarrow \det A = 0$.

Now, as $AA^{-1} = I_n$, $\det(AA^{-1}) = 1$.

$\det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = (\det(A))^{-1}$.

ⓑ Easy to check that $\det E = \det E^t$, if E is elementary. If A is invertible, $A = E_1 \cdots E_k$ for some elementary E_i's. Thus, ~~det~~ $A^t = (E_1 \cdots E_k)^t = E_k^t \cdots E_1^t$

$\Rightarrow \det(A^t) = \det E_k^t \cdots \det E_1^t = \det E_k \cdots \det E_1 = \det A$.

If A is not invertible, so is A^t. This case, $\det A = \det A^t = 0$.

[1] LINEAR TRANSFORMATIONS AND THEIR MATRIX REPRESENTATIONS.

Recall that if $\mathcal{B} = (v_1, \ldots, v_n)$ is an ordered basis of V, then there exists an isomorphism, say $\hat{\mathcal{B}}$: $V \longrightarrow \mathbb{F}^n$, given by

$$v \longmapsto (\alpha_1 \, \alpha_2 \cdots \alpha_n)^t, \text{ where}$$

$(\alpha_1 \, \alpha_2 \cdots \alpha_n)^t$ is the co-ordinate vector of v with respect to $\mathcal{B}$.

On the other hand, let A be a $m \times n$ matrix with entries in $\mathbb{F}$. Then

$$L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m \text{ defined by}$$

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \longmapsto \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m \times n} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^m$$

$$\text{ie. } X \longmapsto AX$$

is a linear transformation ~~of~~ from $\mathbb{F}^n \rightarrow \mathbb{F}^m$.

Let V, W be vector spaces of dim n and m resp., and let $\mathcal{B}, \mathcal{C}$ be bases of V and W resp. Let T be a linear transformation from V to W, and let $A = [T]_{\mathcal{B}}^{\mathcal{C}}$. Then the following diagram commutes.

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^m \\ V \arrow[u, "\hat{\mathcal{B}}"] \arrow[r, "T"'] & W \arrow[u, "\hat{\mathcal{C}}"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, squiggly] & AX \\ v \arrow[u, squiggly] \arrow[r, squiggly] & Tv \arrow[u, squiggly] \end{tikzcd}$$

So, we need to show that if co-ordinate vector of v is X, then the co-ordinate vector of Tv is AX, where A is the matrix of T.

Given $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$, where $X = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

Then $Tv = \alpha_1 T v_1 + \cdots + \alpha_n T v_n$

If $A = (a_{ij})$, then $Tv_j = \sum_1^m a_{ij} w_i$,

where $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{C} = (w_1, \ldots, w_m)$.

$$\begin{aligned}
\text{So, } Tv &= \alpha_1 \left(\sum_1^m a_{i1} w_i \right) + \alpha_2 \left(\sum_1^m a_{i2} w_i \right) + \cdots + \alpha_n \left(\sum_1^m a_{in} w_i \right) \\
&= \left(\sum_1^n \alpha_j a_{1j} \right) w_1 + \left(\sum_1^n \alpha_j a_{2j} \right) w_2 + \cdots + \left(\sum_1^n \alpha_j a_{mj} \right) w_m \\
&= R_1 X \, w_1 + R_2 X \, w_2 + \cdots + R_m X \, w_m
\end{aligned}$$

where R_i is the ith row of A

Thus, the co-ord vector of Tv is $\begin{bmatrix} - R_1 - \\ - R_2 - \\ \vdots \\ - R_m - \end{bmatrix} \begin{bmatrix} | \\ X \\ | \end{bmatrix}$

$= AX$.

Hence, if the co-ord vector of v wrt a basis $\mathcal{B}$ is X, and T is a linear transformation, then the co-ord vector of Tv wrt $\mathcal{C}$ is AX, where $A = [T]_{\mathcal{B}}^{\mathcal{C}}$.

How does the matrix of a linear transformation change, if we change the bases of the vector spaces concerned?

To answer this question, let us try to understand if there is some relation between two (finite) bases of a vector space.

Let V be a vector space of dim n and let $\mathcal{B} = (v_1, \dots, v_n)$ and $\mathcal{B}' = (v_1', \dots, v_n')$ be two bases of V.

We note that every vector in $\mathcal{B}'$ is a linear combination of the vectors in $\mathcal{B}$. Let

$$v_j' = p_{1j} v_1 + p_{2j} v_2 + \dots + p_{nj} v_n .$$

The column vector $P_j = (p_{1j} \ p_{2j} \ \cdots \ p_{nj})^t$ is the co-ordinate vector of v_j' with respect to the basis $\mathcal{B}$. Construct a square matrix P, with P_j's as the column vectors. Then we have the equation of matrices:

$$\mathcal{B}' = \mathcal{B} P .$$

This matrix P is called the base change matrix.

<u>Exercises</u>: ① Show that a base change matrix is invertible.

② If $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V, the other bases are subsets of the form $\mathcal{B}' = \mathcal{B} P$, for P any invertible $n \times n$ matrix.

③ Show that if X is the co-ordinate vector of a $v \in V$, with respect to $\mathcal{B}$, and if $\mathcal{B}'$ is another basis of V, with base change matrix P, then co-ord vector of v with respect to $\mathcal{B}'$ is $P^{-1} X$.

Now, let A be the matrix of T with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W resp. Let $\mathcal{B}'$ and $\mathcal{C}'$ be new bases for V and W resp. By previous discussions, there exists invertible matrices P and Q such that P is the base change matrix of $\mathcal{B}$ and $\mathcal{B}'$, and Q is the base change matrix of $\mathcal{C}$ and $\mathcal{C}'$, ie

$$\mathcal{B}' = \mathcal{B}P \text{ and } \mathcal{C}' = \mathcal{C}Q.$$

Try to prove the following theorem.

Theorem 3.1 : Let A be the matrix of a linear transformation T with respect to given bases $\mathcal{B}$ and $\mathcal{C}$.

(a) Let $\mathcal{B}'$, $\mathcal{C}'$ be new bases such that $\mathcal{B}' = \mathcal{B}P$ and $\mathcal{C}' = \mathcal{C}Q$. The matrix of T with respect to ~~other bases~~ the bases $\mathcal{B}'$, $\mathcal{C}'$ is $A' = Q^{-1}AP$.

(b) The matrices A' which represent T with respect to other bases are those of the form $A' = Q^{-1}AP$, where Q and P are invertible matrices of suitable sizes.

Corollary 3.2 : Let $T : V \to V$ be a linear transformation (called a linear operator as domain and codomain are same). Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for V, and P the base change matrix satisfying $\mathcal{B}' = \mathcal{B}P$. Let A be the matrix of T with respect to $\mathcal{B}$, ie $A = [T]_{\mathcal{B}}^{\mathcal{B}}$.

Then $[T]_{\mathcal{B}'}^{\mathcal{B}'} = P^{-1}AP$.

Also, matrices A' which represent T with respect to other bases are of the form $Q^{-1}AQ$, where Q is an invertible matrix of suitable size.

Definition : Two $n \times n$ matrices A, B are said to be similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

[2] FINDING BASES OF CERTAIN SUBSPACES:

Exercise : Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$.

Find the null space of A. Compute a basis for the null space of A.

Solution : Thinking of A as a real matrix, A determines a linear transformation

$$L_A : \mathbb{R}^4 \to \mathbb{R}^3.$$
$$X \mapsto AX.$$

Null space of A is the null space of L_A.

$$\text{Null space of } A = \{X \in \mathbb{R}^4 : L_A(X) = 0_{\mathbb{R}^3}\}$$
$$= \{X \in \mathbb{R}^4 : AX = 0_{\mathbb{R}^3}\}.$$

Thus, we need to solve the homogenous system of equations

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$2x_1 + x_2 + 4x_3 + 3x_4 = 0$$
$$3x_1 + 4x_2 + x_3 + 2x_4 = 0$$

The row echelon form of A is $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Assigning arbitrary values, say c_1 and c_2 be the variables x_3 and x_4 corresponding to the non-pivotal columns, resp., the equations at this stage look like

$$x_1 + 3c_1 + 2c_2 = 0$$
$$x_2 - 2c_1 - c_2 = 0$$
$$\Rightarrow x_1 = -3c_1 - 2c_2$$
$$x_2 = 2c_1 + c_2.$$

Thus, elements of the null space look like

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3c_1 - 2c_2 \\ 2c_1 + c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

This shows that the null space is spanned by the set $\mathcal{B} := \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The set $\mathcal{B}$ is also linearly independent. Why?

Thus $\mathcal{B}$ forms a basis for the null space of A.

— x —

Given a $m \times n$ matrix A, the range of A (also called the column space of A) is the range of the linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$, where entries of A are from $\mathbb{F}$.

Note that if $A_1, \ldots, A_n$ are the columns of A, range or column space is $\operatorname{span}(\{A_1, \ldots, A_n\})$; but $\{A_1, \ldots, A_n\}$ may not be a linearly independent subset of $\mathbb{F}^m$.

Thus, $\{A_1, \dots, A_n\}$ may not form a basis of range.
For example, range (or column space) of $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ is one-dimensional, as $A_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ are linearly dependent.

In case, $\{A_1, \dots, A_n\}$ is a linearly dependent set, a proper subset of it forms a basis of range of A. How do we find this subset?

Fortunately, elementary row operations do not effect the dependence relations between column vectors. Thus, we use row reductions to find a basis for range(A).
The columns corresponding to pivotal columns in RREF gives a basis for range(A).

<u>Exercise</u>: Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$.

Find the range of A. Compute a basis for the ~~column space of A~~ range of A. What is the rank of A?

<u>Solution</u>: Since the row echelon form of A is $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we see that column 3 and column 4 are linear combinations of column 1 and column 2.

$$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

~~Thus, non pivotal~~ Non-pivotal columns can be written as linear combinations of pivotal columns, and pivotal columns form a linearly independent set of vectors. The corresponding columns in the matrix A form a basis for range(A).

Thus, $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\1\\4\end{bmatrix}$ form a basis for range(A).

Thus rank (A) = 2.

Verify rank-nullity theorem.

Exercise : Prove that a square matrix A is invertible if and only if its columns are linearly independent. What about rows ?

Definition : Let A be a $m \times n$ matrix over $\mathbb{F}$. Row space of A is the subspace of $\mathbb{F}^n$, spanned by the rows of A. So, the set of row vectors of A forms a spanning set of the row space of A.

But the row vectors may not be linearly independent. In order to find a basis for the row space, note that, the row space doesn't change with row operations. Thus, the non-zero row vectors of the row reduced echelon form gives us a basis for the row space.

[3] LINEAR OPERATORS

A linear operator is a linear transformation $T: V \to V$, mapping a vector space to itself.

Example : Let $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. Then,

$$L_{R_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{bmatrix}a\\b\end{bmatrix} \to R_\theta \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\cos\theta - b\sin\theta\\ a\sin\theta + b\cos\theta\end{bmatrix}$$ is a linear operator on $\mathbb{R}^2$.

R_θ is called the rotation matrix. This operator rotates the plane anticlockwise by an angle θ.

Proposition 3.3 : Let T be a linear operator on a finite dimensional vector space V. Then the following are equivalent :

ⓐ T is bijective

ⓑ Kernel (T) or $N(T)$ is the trivial subspace $\{0_V\}$.

ⓒ Range (T) or $R(T)$ is V.

Proof: Exercise.

[4] EIGENVALUES, EIGENVECTORS OF LINEAR OPERATORS.

Definition : Let $T: V \to V$ be a linear operator. An eigenvector v of ~~a linear~~ T is a non-zero vector in V such that $Tv = \lambda . v$, for some ~~sa~~ scalar λ, ie for some $\lambda \in \mathbb{F}$. A non-zero vector in $\mathbb{F}^n$ is an eigenvector of a square $n \times n$ matrix A if it is an eigenvector of the operator $L_A : \mathbb{F}^n \to \mathbb{F}^n$.

Thus $X = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^n$ is an eigenvector of A if

$L_A X = \lambda X$ for some $\lambda \in \mathbb{F}$. In other words,

$AX = \lambda X$, for some $\lambda \in \mathbb{F}$.

Exercise : Show that eigenvectors of T are same as the eigenvectors of the matrices representing T.

The scalar λ that appears in the relation $Tv = \lambda v$ is called the eigenvalue corresponding to the eigenvector v.

Note that :

- Eigenvectors are non-zero vectors.
- Eigenvalues can be zero.
- If v is an eigenvector corresponding to the zero eigenvalue, then $Tv = 0.v = 0_V$
  $\Rightarrow v \in$ Kernel of T, ie the null-space of T.
- If $v \in V$ is an eigenvector, any $\alpha.v$ is also an eigenvector, for any non-zero $\alpha \in \mathbb{F}$, corresponding to the same eigenvalue as v :

$$
\begin{aligned}
Tv = \lambda.v, \text{ say. Then } T(\alpha.v) &= \alpha.T(v) \\
&= \alpha.(\lambda.v) \\
&= (\alpha\lambda).v \\
&= (\lambda\alpha).v \quad (\mathbb{F} \text{ is comm. wrt mult.}) \\
&= \lambda.(\alpha.v)
\end{aligned}
$$

- Eigenvectors corresponding to eigenvalue 1 are fixed by T.
- If $V = \mathbb{R}^n$, a non-zero vector v is an eigenvector if v and $T(v)$ are parallel.
- If $v \in V$ is a non-zero vector and W is the subspace generated by v; let $T(W) \subseteq W$. Then v is an eigenvector of V.

How do we calculate eigenvalues and eigenvectors of a linear operator T?

Recall that an eigenvector of a linear operator T is a nonzero vector v such that $Tv = \lambda v$, for some $\lambda \in \mathbb{F}$.

Note that if T is an operator on V, then

$$\lambda I - T : V \to V \text{ is also a linear operator,}$$

given by $(\lambda I - T)(v) = \lambda v - Tv$, $\forall\, v \in V$, where I is the identity operator.

Hence, a non-zero vector v is an eigenvector with eigenvalue λ

$\iff Tv = \lambda v \quad , \quad v \neq 0$

$\iff (\lambda I - T)v = 0 \quad , \quad v \neq 0$

$\iff v \in \text{Kernel}\,(\lambda I - T) \quad , v \neq 0$

$\iff$ Null space of $(\lambda I - T) \neq \{0_V\}$.

<u>Definition</u>: An operator $T : V \to V$ is said to be singular if its null space is non-trivial, ie null space is not $\{0_V\}$.

<u>Proposition 3.4</u>

Let T be a linear operator on a finite dim. vector space V. The following conditions are equivalent:

ⓐ T is a singular operator

ⓑ T has an eigenvalue equal to zero

ⓒ If A is the matrix of T wrt an arbitrary basis, then $\det A = 0$.

ⓐ $\Rightarrow$ ⓑ

Let T be singular

$\Rightarrow 0 \neq v \in N(T)$

$\Rightarrow Tv = 0_V = 0.v$

$\Rightarrow 0$ is an eigenvalue of T.

ⓑ $\Rightarrow$ ⓒ

Let A be the matrix of with respect to a basis $\mathcal{B}$.

If 0 is an eigenvalue of T, $\exists\ 0 \neq v \in V$, such that ~~A~~ $Tv = 0_V$.

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^n \\ V \arrow[u, "\hat{\mathcal{B}}"] \arrow[r, "T"'] & V \arrow[u, "\hat{\mathcal{B}}"'] \end{tikzcd}$$

~~Since~~ Considering the commutative diagram, if X is the co-ord vector of v wrt $\mathcal{B}$, then

$$\begin{tikzcd} X \arrow[r, squiggly, "L_A"'] & 0 \\ v \arrow[u, squiggly, "\hat{\mathcal{B}}"] \arrow[r, squiggly, "T"'] & 0 \arrow[u, squiggly, "\hat{\mathcal{B}}"'] \end{tikzcd}$$

$L_A : \mathbb{F}^n \to \mathbb{F}^n$ is the operator induced by A, then $L_A(X) = 0$. As $v \neq 0$, $X \neq 0$.

$\Rightarrow AX = 0$, for $X \neq 0$

$\Rightarrow A$ is not invertible.

$\Rightarrow \det A = 0$.

ⓒ $\Rightarrow$ ⓐ. If $\det A = 0$.

$\Rightarrow A$ is not invertible

$\Rightarrow \exists$ a non-zero solution, say $X_0 = (\alpha_1, \dots, \alpha_n)$, of the homogeneous system $AX = 0$.

$\Rightarrow 0 \neq v = \alpha_1 v_1 + \dots + \alpha_n v_n$ is such that $Tv = 0$, where $A = [T]_{\mathcal{B}}^{\mathcal{B}}$, and $\mathcal{B} = (v_1, \dots, v_n)$

$\Rightarrow T$ is singular.

Check that if A is the matrix of T, ~~then~~ with respect to some basis $\mathcal{B}$, then matrix of the operator $(\lambda I - T)$ is $(\lambda I - A)$, with respect to $\mathcal{B}$.

Thus, we say that a non-zero vector v is an eigen-vector, with eigenvalue λ, of T iff $\det(\lambda I - A) = 0$.

Now, $\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & \cdots & & \lambda - a_{nn} \end{bmatrix}$

and $\det(\lambda I - A)$ is a polynomial of degree n, in the ~~v~~ indeterminate λ.

<u>Definition</u> : The characteristic polynomial of a linear operator T is the polynomial

$$p(x) = \det(xI - A),$$

where A is the matrix of T with respect to some basis.

Thus, eigenvalues of T are roots of the characteristic polynomial $p(x)$.

As
$$\begin{aligned} \det(xI - A) &= \det I_n \det(xI - A) \\ &= \det(P^{-1}P)\det(xI - A) \\ &= \det(P^{-1}) \det P \det(xI - A) \\ &= \det(P^{-1}) \det(xI - A) \det P \\ &= \det(P^{-1}(xI - A)P) \\ &= \det(xI - P^{-1}AP), \end{aligned}$$

the characteristic polynomial does not depend on the basis.

Exercise: Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear operator that is defined by $T(f(x)) = f(x) + x f'(x) + f'(x)$, and let $\mathcal{B} = (1, x, x^2)$ be the standard basis of $P_2(\mathbb{R})$. ~~Then~~ Find the eigenvalues of T.

The matrix of T, with respect to $\mathcal{B}$ is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$ as, $T(1) = 1$
$T(x) = 2x+1$
$T(x^2) = 3x^2 + 2x$

Characteristic polynomial ~~with respect to~~ of T is $\det \begin{bmatrix} x-1 & -1 & 0 \\ 0 & x-2 & -2 \\ 0 & 0 & x-3 \end{bmatrix}$

So eigenvalues are roots of $\det \begin{bmatrix} x-1 & -1 & 0 \\ 0 & x-2 & -2 \\ 0 & 0 & x-3 \end{bmatrix} = 0$

ie $(x-1)(x-2)(x-3) = 0$

$\Rightarrow$ eigenvalues are $\lambda = 1, 2$ and 3.

Let us try to find the corresponding eigenvectors:

Let $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to $\lambda = 1$.

As, $(\lambda I - A)v = 0$, for $\lambda = 1$, we have,

Then $\begin{bmatrix} 1-1 & -1 & 0 \\ 0 & 1-2 & -2 \\ 0 & 0 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing $\begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$ to row echelon form, we get

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Assigning arbitrary value c to the non-pivotal variable x_1, the solution set is $c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So, all eigenvectors of $\lambda = 1$ are of the form $c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $c \neq 0$.

Find the eigenvectors corresponding to eigenvalues $\lambda = 2$ and 3.