LECTURE - 2

A field $\mathbb{F}$ for us shall denote $\mathbb{R}$ or $\mathbb{C}$.

[1] VECTOR SPACES

Definition: A vector space V over ~~the~~ a field $\mathbb{F}$ consists of a set V, and two functions, denoted by $+$ and $\cdot$,

$$+ : V \times V \to V$$
$$\cdot : \mathbb{F} \times V \to V$$

Such that, (denoting $+(v_1, v_2) = v_1 + v_2 \in V$, and $\cdot(c, v) = c.v \in V$)

1) $v_1 + v_2 = v_2 + v_1$, $\forall\, v_1, v_2 \in V$
2) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$, $\forall\, v_1, v_2, v_3 \in V$
3) $\exists$ an element $0_v \in V$ such that $0_v + v = v + 0_v = v$, $\forall\, v \in V$. (0_v is the additive identity)
4) For each $v \in V$, $\exists$ an element $(-v)$ in V, such that $v + (-v) = 0_v$. ($-v$ is called the additive identity of v).
5) $1.v = v$, $\forall\, v \in V$.
6) $(\alpha\beta).v = \alpha.(\beta.v)$ $\forall\, \alpha, \beta \in \mathbb{F}$, and $v \in V$.
7) $\alpha.(v_1 + v_2) = \alpha.v_1 + \alpha.v_2$ $\forall\, \alpha \in \mathbb{F}$, $v_1, v_2 \in V$.
8) $(\alpha + \beta)v = \alpha v + \beta v$, $\forall\, \alpha, \beta \in \mathbb{F}$ and $v \in V$.

Elements in a vector space are called vectors, and elements in a field (here $\mathbb{R}$ or $\mathbb{C}$) are called scalars.

Examples :

1. Let $\mathbb{F}$ be a field, and let $\mathbb{F}^n$ denote the set of all n-tuples with entries from $\mathbb{F}$, ie

$$\mathbb{F}^n = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \ \alpha_i \in \mathbb{F}, \ 1 \le i \le n \right\}, \quad n \ge 1.$$

$\mathbb{F}^n$ is a vector space over $\mathbb{F}$, for all $n \ge 1$, with the following operations:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix}$$

here, the addition is addition in the field, $\mathbb{R}$ or $\mathbb{C}$.

and $$\alpha \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha\alpha_1 \\ \alpha\alpha_2 \\ \vdots \\ \alpha\alpha_n \end{bmatrix}$$

here, the multiplication is the multiplication in $\mathbb{R}$ or $\mathbb{C}$.

2. The set of all $m \times n$ matrices with entries from $\mathbb{F}$, is a vector space, denoted by $M_{m\times n}(\mathbb{F})$. Operations addition (of two $m\times n$ matrices) and scalar multiplication have already been defined in LECTURE NOTE 1.

3. The set of all polynomials with coefficients in $\mathbb{F}$ is a vector space denoted by $P(\mathbb{F})$.

Ex 4. Proposition 2.1 Let V be a vector space over $\mathbb{F}$.

(a) for $v_1, v_2, v_3 \in V$, $v_1 + v_3 = v_2 + v_3$ implies $v_1 = v_2$.

- $v_1 + v_3 = v_2 + v_3$

$\Rightarrow (v_1 + v_3) + (-v_3) = (v_2 + v_3) + (-v_3)$

$\Rightarrow v_1 + (v_3 + (-v_3)) = v_2 + (v_3 + (-v_3))$

$\Rightarrow v_1 = v_2$.

(b) 0_V is unique in V.

(c) Given $v \in V$, $(-v)$ is unique in V.

(d) $0 . v = 0_V$, $\forall v \in V$

(e) $(-\alpha) . v = -(\alpha . v) = \alpha . (-v)$, $\forall \alpha \in \mathbb{F}$ and $\forall v \in V$.

(f) $\alpha . 0_V = 0_V$, $\forall \alpha \in \mathbb{F}$.

[2] SUBSPACES

Whenever we study an algebraic structure, it is of interest to study subsets which possess the same structure.

Definition : A subset W of a vector space V over a field $\mathbb{F}$ is called a subspace of V if W is a vector space over $\mathbb{F}$ under the operations of addition and scalar multiplication defined on V.

Theorem : Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V:

ⓐ $0_V \in W$;

ⓑ $\forall\, v_1, v_2 \in W,\ v_1 + v_2 \in W$; (W is closed under addition)

ⓒ $\forall\, \alpha \in \mathbb{F},\ \forall\, v \in W,\ \alpha v \in W$; (W is closed under scalar multiplication)

Proof : Let W be a subspace of V. Then ⓑ and ⓒ hold. Also, $\exists\, 0_W \in W$ such that

$0_W + v = v,\ \forall\, v \in W$.

But $0_V + v$ is also v.

Hence, by cancellation law, prop. 2.1 (a),

$0_V = 0_W$.

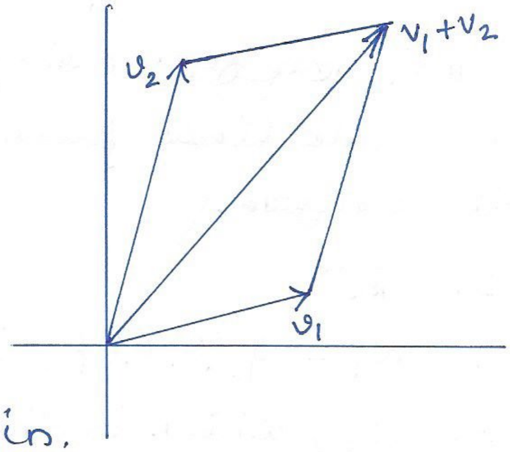
Conversely, let W be a subset of V, such that the three conditions ⓐ, ⓑ and ⓒ are satisfied. To show W is a subspace, we only need to show that additive inverse of every element in W, belongs to W. Now if $v \in W$, $(-1).v \in W$ by ⓒ.

But $-v = (-1).v$ by proposition 2.1 (e). Hence, W is a subspace of V.

Examples

1. Find all subspaces of $\mathbb{R}^2$.

Except for $\mathbb{R}^2$ itself and the zero subspace $W = \{(0,0)\}$, the only non-trivial subspaces are ~~lines~~ straight lines through the origin.



2. Let A be a symmetric matrix, ie $A = A^t$. The set of all $n \times n$ symmetric matrices W form a subspace of $M_{n\times n}(\mathbb{F})$.

Verify that :

ⓐ Zero matrix $\in W$

ⓑ If $A, B \in W$, then $A+B \in W$, as
$$(A+B)^t = A^t + B^t = A+B$$

ⓒ If $\alpha \in \mathbb{F}$ and $A \in W$, then $\alpha A \in W$, as
$$(\alpha A)^t = \alpha A^t = \alpha A.$$

3. Let $P_n(\mathbb{F})$ denote the set of all polynomials over $\mathbb{F}$, of degree less than or equal to n. Then $P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$.

Convince yourself that any intersection of subspaces of a vector space V is a subspace of V.

Exercises :

Definition : Let X_1, X_2 be non-empty subsets of a vector space V. The sum of X_1 and X_2, denoted by $X_1 + X_2$, is the set $\{v_1 + v_2 : v_1 \in X_1, v_2 \in X_2\}$.

1. Prove that if W_1, W_2 are subspaces of a vector space V, then $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2. Also, prove that $W_1 + W_2$ is the smallest subspace that contains both W_1 and W_2.

Definition : A vector space V is said to be the direct sum of W_1 and W_2, denoted by $V = W_1 \oplus W_2$, if W_1 and W_2 are subspaces of V, such that $W_1 \cap W_2 = \{0_V\}$ and $W_1 + W_2 = V$.

2. Show that $\mathbb{F}^n$ is the direct sum of the subspaces $W_1 = \left\{ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^n : \alpha_n = 0 \right\}$, and

$$W_2 = \left\{ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^n ; \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0 \right\}.$$

[3] LINEAR COMBINATIONS

Definition : Let V be a vector space and S be a non-empty subset of V. A vector $v \in V$ is said to be a linear combination of elements of S if there exists a finite number of vectors $v_1, v_2, \dots, v_n$ in S, and scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ such that

$$v = \alpha_1 . v_1 + \alpha_2 . v_2 + \dots + \alpha_n . v_n .$$

We say, v is a linear combination of $v_1, \dots, v_n$.

Exercise :

1. Determine if the vector $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ is a linear combination of the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $v_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $v_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$.

Solution : We need to determine if there exist scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \alpha_5 \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$$

So $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ can be written as a linear combination of v_1, v_2, v_3, v_4, v_5, if and only if there exists a non-trivial solution of the system of equations

$$\begin{aligned} x_1 \quad - 2x_2 \qquad\quad + 2x_4 - 3x_5 &= 2 \\ 2x_1 - 4x_2 + 2x_3 \qquad\quad + 8x_5 &= 6 \\ x_1 - 2x_2 + 3x_3 - 3x_4 + 16x_5 &= 8 \end{aligned}$$

Row reduced echelon form of the augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{array}\right]$$

Note that, 2nd column and 5th column do not contain pivots.

Assigning arbitrary value c_1 to x_2 and c_2 to x_5, the equations read:

$$x_1 - 2c_1 + c_2 = -4$$

$$x_3 + 3c_2 = 7$$

$$x_4 - 2c_2 = 3.$$

$$\Rightarrow \quad x_3 = 7 - 3c_2, \quad x_4 = 3 + 2c_2, \quad x_1 = -4 + 2c_1 - c_2.$$

So, every solution is of the form:

$$\cancel{\begin{bmatrix} -4 \\ 0 \\ 7 \\ 3 \\ 0 \end{bmatrix}} + c_1 \cancel{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}} + c_2 \cancel{\begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \\ -1 \end{bmatrix}},$$ ~~for some values of c_1 and c_2.~~

~~In particular, for $c_1 = c_2 =$~~

$$\begin{bmatrix}-4\\0\\7\\3\\0\end{bmatrix} + c_1\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + c_2\begin{bmatrix}-1\\0\\-3\\2\\1\end{bmatrix}$$, for some values of c_1 and c_2.

In particular, for $c_1 = c_2 = 0$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-4, 0, 7, 3, 0)$ is a solution.

Theorem 2.2 If S is a non-empty subset of a vector space V, then the subset W consisting of all linear combinations of elements of S is a subspace of V. Moreover, W is the smallest subspace of V containing S, ie. W is a subset of any subspace of V ~~so~~ that contains S.

Proof. Exercise. □.

Definition : The subspace W described in Theorem 2.2 is called the span of S, and is denoted by span(S). By convention, span$(\Phi) = \{0_V\}$.

Definition : A subset S of V is said to span (or generate) V if span $S = V$.

Example : The set $S = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

generate $\mathbb{R}^3$, as any vector $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination of elements in S, by: ~~$(a_1, 0$~~

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where $c_1 = \frac{1}{2}(a_1 + a_2 - a_3)$, $c_2 = \frac{1}{2}(a_1 - a_2 + a_3)$, and $c_3 = \frac{1}{2}(-a_1 + a_2 + a_3)$.

Definition : A subset S of a vector space V is said to be linearly dependent if there exist a finite number of distinct vectors $x_1, x_2, \dots, x_n$ in S and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$, not all zeros, such that

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0 .$$

Example : The set $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

is a linearly dependent subset of $\mathbb{R}^4$, as

$$4 \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix} + (-3) \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition : A subset S of a vector space that is not linearly dependent is said to be linearly independent.

- Empty set is lin. ind.
- Set consisting of a single non-zero vector is linearly independent.

[4] BASIS AND DIMENSION

Definition : A basis $\mathcal{B}$ for a vector space V is a linearly independent subset of V, that spans V.

Examples ① Let $S = \left\{ e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{-- } i \; ; \; 1 \le i \le n \right\} \subseteq \mathbb{F}^n$.

Then, one can verify readily ~~that~~ $\text{Span}(S) = \mathbb{F}^n$, and S is linearly independent.

Thus, S is a basis for $\mathbb{F}^n$, called the standard basis of $\mathbb{F}^n$.

② $P_n(\mathbb{F})$ is vector space of all polynomials over $\mathbb{F}$, of degree ~~less than~~ or equal to n.
The set $S = \{1, x, x^2, \ldots, x^n\}$ forms a basis of $P_n(\mathbb{F})$.

③ For $P(\mathbb{F})$, the vector space of all polynomials over $\mathbb{F}$, $S = \{1, x, x^2, \ldots\}$ is a basis.

Theorem 2.3 : Let V be a vector space and $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a subset of V. Then $\mathcal{B}$ is a basis for V if and only if each vector $v \in V$ can be uniquely expressed as a linear combination of vectors in $\mathcal{B}$.

Proof : Exercise $\square$

If we fix an ordering of the set $\mathcal{B} = \{v_1, \dots, v_n\}$ then theorem 2.3 suggests that every $v \in V$ determines a unique n-tuple of scalars $(\alpha_1, \dots, \alpha_n)$. Conversely, each n-tuple of scalars uniquely determines a vector $v \in V$. Thus V 'looks like' $\mathbb{F}^n$, where n is the number of elements in the basis $\mathcal{B}$.

<u>Theorem 2.4</u>: Let S be a linearly independent subset of a vector space V, and let $v \in V \setminus S$. Then, $S \cup \{v\}$ is also linearly independent if and only if $v \notin \text{Span}(S)$.

<u>Proof</u>: ~~Let $S = \{v_1, \dots, v_n\}$~~

If $S \cup \{v\}$ is linearly dependent, then there exist vectors $v_1, \dots, v_n$ in $S \cup \{v\}$, and ~~non~~ not-all-zero scalars $\alpha_1, \dots, \alpha_n$ such that $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$. As S is lin. ind., one of the v_i's must be v, say v_1. Also, $\alpha_1 \neq 0$, otherwise S becomes lin. dep. Thus,

$$v = v_1 = \alpha_1^{-1}(-\alpha_2 v_2 - \dots - \alpha_n v_n) \in \text{Span}(S).$$

Conversely, if $v \in \text{Span } S$, then

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n, \quad v_i \in S, \ \alpha_i \in \mathbb{F}.$$

$$\Rightarrow 0 = (-1)v + \alpha_1 v_1 + \dots + \alpha_n v_n.$$

$\Rightarrow \{v, v_1, \dots, v_n\} = S \cup \{v\}$ is a linearly dependent set.

Theorem 2.5 If a vector space V is generated by a finite set S_0, then a subset of S_0 is a basis for V. Hence, V has a finite basis.

Proof. If S_0 is the empty set, or $\{0_V\}$, then $V = \{0_V\}$, and the empty set is a basis for V. Let S_0 contain a non-zero vector v_1. Then $\{v_1\}$ is lin. ind. By previous theorem, we continue extending $\{v_1\}$ to $\{v_1, v_2, \dots, v_r\}$, $v_i \in S_0$, $2 \le i \le r$, such that $\{v_1, \dots, v_r\}$ is lin. ind. As S_0 is finite, this process should terminate after finitely many steps, ie we should be able to get

$S = \{v_1, \dots, v_r\} \subseteq S_0$, S lin. ind, and any extension of S by an element of S_0 would give a lin. dep. set. We show S is a basis of V. Enough to show that $\text{Span}(S) = V$. We show that $S_0 \subseteq \text{span}(S)$. Let $v_0 \in S_0$. If $v_0 \in S$, then $v_0 \in \text{span}\, S$. If $v_0 \notin S$, then $S \cup \{v_0\}$ is lin. dep.

$\Rightarrow x \in \text{span}\, S$, by theorem 2.4.

$\Rightarrow S_0 \subseteq \text{span}\, S$.

Thus, $V = \text{span}\, S_0 \subseteq \text{span}\, S$

$\Rightarrow V = \text{span}\, S$.

Theorem 2.6

Let V be a vector space which has a finite basis.

(a) Let S be a finite subset that spans V, and let L be a lin. independent subset of V. One can obtain a basis of V by adding elements of S to L

(b) Let S be a finite subset that spans V. One can obtain a basis of V by deleting elements from S.

Proof.
Exercise. $\square$

Theorem 2.7

Let S and L be finite subsets of a vector space V. Assume $V = \text{span}\, S$ and L is lin. independent. Then S contains atleast as many elements as L does, ie $|S| \geq |L|$.

$\square$

Proof: Refer to Artin

Proposition 2.8. Let V have a finite basis.

(a) Any two bases of V have the same order,

(b) Let $\mathcal{B}$ be a basis. If $\text{span}\, S = V$, then $|S| \geq |\mathcal{B}|$ and $|S| = |\mathcal{B}|$ if and only if S is a basis.

(c) Let $\mathcal{B}$ be a basis. If a set L of vectors is lin. independent, then $|L| \leq |\mathcal{B}|$ and $|L| = |\mathcal{B}|$ if and only if L is a basis.

Definition: V is called finite dimensional if it has a finite basis, say $\mathcal{B}$, and $|\mathcal{B}|$ is called the dimension of V, denoted by dim V.

[5] LINEAR TRANSFORMATIONS

Definition : Let V, W be vector spaces over a field $\mathbb{F}$. A function $T: V \to W$ is called a linear transformation from V to W if for all $v_1, v_2 \in V$, and $\alpha \in \mathbb{F}$,

ⓐ $T(v_1 + v_2) = T(v_1) + T(v_2)$,

ⓑ $T(c v_1) = c T(v_1)$.

A bijective linear transformation is called an isomorphism.

Examples :

1. Let $V = P_n(\mathbb{R})$ and $W = P_{n-1}(\mathbb{R})$.
$T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ given by
$$f \longmapsto f', \text{ the derivative of } f,$$
is a linear transformation.

2. Let $V = C(\mathbb{R})$, the real vector space of continuous real-valued functions on $\mathbb{R}$. Let $a, b \in \mathbb{R}$, $a < b$. Define $T: V \to \mathbb{R}$ by
$$T(f) = \int_a^b f(t)\,dt,$$
$\forall f \in V$. Then, verify ~~that~~ T is a linear transformation.

3. $T: V \to V$
$v \longmapsto v \quad \forall v \in V$ is the identity linear transformation, and
$T: V \to V$
$v \longmapsto 0_V, \ \forall v \in V$ is the zero transformation,

Exercise: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 + a_2 \\ a_1 \end{bmatrix}$.

Show that T is a linear transformation.

Definitions : Let $T : V \to W$ be a linear transformation. We define null space (or kernel) of T to be

$$N(T) = \{ v \in V \; : \; Tv = 0_W \} \subseteq V.$$

We define range (or image) of T to be the subset of W, which is the image of T, ie

$$R(T) = \{ T(v) : v \in V \} \subseteq W.$$

Exercise : Prove that null space is a subspace of V and range is a subspace of W.

Definitions : Let V, W be vector spaces, and $T : V \to W$ a linear transformation. If $N(T)$, $R(T)$ are finite dimensional, then we define

nullity $(T) = \dim N(T)$ and

rank $(T) = \dim R(T)$.

Theorem 2.9 (Rank-Nullity theorem or dimension theorem).

Let V, W be vector spaces, and let $T : V \to W$ be linear. If V is finite dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim V.$$

Proof : Suppose $\dim V = n$ and let $\{v_1, \ldots, v_k\}$ be a basis for $N(T)$. We can extend $\{v_1, \ldots, v_k\}$ to a basis $\mathcal{B} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V.

We show that $\mathcal{C} = \{Tv_{k+1}, \ldots, Tv_n\}$ is a basis for $R(T)$. Let $w \in R(T)$. Then $w = T(v)$ for some $v \in V$. There exists $c_1, \ldots, c_n$ such that

$$v = c_1 v_1 + \cdots + c_k v_k + c_{k+1} v_{k+1} + \cdots + c_n v_n$$

$$\begin{aligned} w = T(v) &= T(c_1 v_1 + \cdots + c_k v_k + c_{k+1} v_{k+1} + \cdots + c_n v_n) \\ &= T(c_1 v_1) + \cdots + T(c_k v_k) + T(c_{k+1} v_{k+1}) + \\ &\qquad \cdots + T(c_n v_n) \\ &= c_1 T v_1 + \cdots + c_k T v_k + c_{k+1} T v_{k+1} + \\ &\qquad \cdots + c_n T v_n \\ &= c_{k+1} T v_{k+1} + \cdots + c_n T v_n, \\ &\qquad \text{as } Tv_1 = \cdots = Tv_k = 0. \end{aligned}$$

$\Rightarrow w \in \operatorname{Span} \mathcal{C}$.

$\Rightarrow \operatorname{span} \mathcal{C} = R(T)$.

Now, to show $\mathcal{C}$ is linearly independent, let $\exists\ \alpha_{k+1}, \ldots, \alpha_n \in \mathbb{F}$ such that

$$\alpha_{k+1} T v_{k+1} + \cdots + \alpha_n T v_n = 0_W. \qquad \text{— (1)}$$

$$\Rightarrow T(\alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n) = 0_W$$

$$\Rightarrow \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n \in N(T)$$

$$\Rightarrow \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_k v_k$$

for some β_i's $\in \mathbb{F}$.

Since $\mathcal{B}$ is a basis, $\alpha_{k+1} = \cdots = \alpha_n = \beta_1 = \cdots = \beta_k = 0$. ∎

Corollary 2.10 : Let V, W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be a linear transformation. Then T is one-one if and only if T is onto.

Proof: Note that T is one-one if and only if $N(T) = \{0_V\}$.

[6] MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION.

Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis for V. Let $v \in V$. Then $\exists$ a unique n-tuple $(\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{F}$, such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

The vector $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^n$ is called the co-ordinate vector of v, with respect to $\mathcal{B}$. We denote it by $[v]_{\mathcal{B}}$.

Example : Let $V = P_2(\mathbb{R})$, and let $\mathcal{B} = (1, x, x^2)$. If $f(x) = 4x^2 + 1$, then $[f]_{\mathcal{B}}$ is $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$.

Suppose that V and W are finite-dimensional vector spaces with ordered bases $\mathcal{B} = (v_1, \dots, v_n)$ and $\mathcal{C} = (w_1, \dots, w_m)$ respectively. Let $T: V \to W$ be a linear transformation. Then there exist unique scalars $\alpha_{ij} \in \mathbb{F}$, $1 \le i \le m$, $1 \le j \le n$, such that $Tv_j = \sum_{i=1}^{m} \alpha_{ij} w_i$, $\forall\ 1 \le j \le n$.

<u>Definition</u> : With the above notations, the matrix ~~A = (αij)~~ A, whose (i,j)th entry is α_{ij}, $1 \le i \le m$, $1 \le j \le n$, is called the matrix of the linear transformation T, with respect to $\mathcal{B}$ and $\mathcal{C}$, call it $[T]_{\mathcal{B}}^{\mathcal{C}}$. If $V = W$ and $\mathcal{B} = \mathcal{C}$, then we write $A = [T]_{\mathcal{B}}$.

Note that the jth column of the matrix of T is the co-ordinate vector of the vector Tv_j, with respect to $\mathcal{C}$.

<u>Example</u> : Define $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by
$$f \longmapsto f'.$$
Let $\mathcal{B} = (1, x, x^2, x^3)$ and $\mathcal{C} = (1, x, x^2)$.
Then, the matrix is T is given by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Note that :

$T(1) = 0 = 0.1 + 0.x + 0.x^2$

$T(x) = 1 = 1.1 + 0.x + 0.x^2$

$T(x^2) = 2x = 0.1 + 2.x + 0.x^2$

$T(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2$

Verify the following :

Proposition : Let V, W be finite dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$ resp. Let $T, S : V \to W$ be linear transformations. Then,

i) $[T+S]_{\mathcal{B}}^{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} + [S]_{\mathcal{B}}^{\mathcal{C}}$

ii) $[\alpha T]_{\mathcal{B}}^{\mathcal{C}} = \alpha [T]_{\mathcal{B}}^{\mathcal{C}}$, where $\alpha T : V \to W$ is given by $(\alpha T)(v) = \alpha . T(v)$, $\forall \alpha \in \mathbb{F}$.

Proposition : Let V, W, Z be finite dimensional vector spaces with ordered bases $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ resp. ~~Then~~ Let $S : V \to W$ and $T : W \to Z$ be linear transformations. Then $TS : V \to Z$ defined by $(TS)(v) = T(S(v))$ is also a linear transformation and

$$[TS]_{\mathcal{A}}^{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [S]_{\mathcal{A}}^{\mathcal{B}}.$$

Here, right hand side is product of matrices, $[T]_{\mathcal{B}}^{\mathcal{C}}$ is of order $p \times m$, and $[S]_{\mathcal{A}}^{\mathcal{B}}$ is of order $m \times n$, where $\dim V = n$, $\dim W = m$ and $\dim Z = p$.

Exercise : Show that if P is an invertible $n \times n$ matrix, then $L_P : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism.