Download codes for this lecture at:

Lecture 5 codes: https://www.dropbox.com/s/zay3xqqbfhggmj6/Lec5.R?dl=0

Recap

Recall that if $X \sim \text{Uniform}(a, b)$, then

$$\Pr(a_1 < X < b_1) = \frac{b_1 - a_1}{b - a} =: p\,.$$

Recall that we obtained a method to estimate p by $\hat{p}$, by uniformly obtaining points on the line (a, b), and calculating the proportion of points in (a_1, b_1). Additionally, an estimate of the length of the interval (a_1, b_1) is

$$(b - a)\hat{p}\,.$$

What if we wanted to estimated the area of a region instead of a length of a segment?

Area of a Circle

Consider a unit circle centered at $(0, 0)$:

$$x^2 + y^2 < 1\,.$$

We are interested in *estimating* the area of a circle. That is, we are interested in estimating

$$\theta = \int\int \mathbb{I}(x^2 + y^2 < 1)dxdy\,.$$

Suppose $Z = (X, Y)$ is uniformly distributed on a square. That is, all points within the square are equally likely to be picked. We know how to sample uniformly from a square.

Let $X \sim \text{Uniform}(a, b)$ and $Y \sim \text{Uniform}(a, b)$. Then (X, Y) are uniformly distributed over the square $(a, b) \times (a, b)$. To convince yourself of this, run the code below a few times

```r
a <- 0
b <- 1
n <- 1e4
x <- runif(n, min = 0, max = 1)
y <- runif(n, min = 0, max = 1)

plot(x, y)
```

If we can enclose our circle inside of a square box, obtain points uniformly in the box, and check how many lie inside the circle. Let $p = \Pr((X,Y)$ lie in the circle is), than

$$\hat{p} = \frac{\text{Number of points in the circle}}{\text{Number of points drawn}}$$

And thus, as before, the area of the circle can be obtained by

$$\text{Area of circle} = \hat{p} \cdot \text{Area of the box}\,.$$

```
n <- 1e4
xvec <- runif(n, min = -1, max = 1) #c(U1, U2)
yvec <- runif(n, min = -1, max = 1) #c(U1, U2)
in.or.out <- (xvec^2 + yvec^2 < 1)
mean(in.or.out)*4

# graph of points that are and out
plot(xvec, yvec, xlab = "x", ylab = "y", main = "In or Out", asp = 1, col =
    in.or.out+1)
```

Volume of higher-dimensional spheres

We know the area of a circle, and we know the volume of the sphere, but of course, we don't know the volume of a general k-dimensional sphere. Consider the k-sphere

$$x_1^2 + x_2^2 + \cdots + x_k^2 < 1$$

Then the volume of the k-sphere is

$$\int \mathbb{I}(x_1^2 + x_2^2 + \cdots + x_k^2 < 1) dx_1 \dots dx_k\,.$$

We can repeat the same idea now. Enclose the k-sphere in a k-box, obtain $\hat{p}$, the proportion of points in the region and estimate the volume of the sphere.

$$\text{Volume of } k\text{-sphere} = \hat{p} \cdot \text{Volume of the } k\text{-box}\,.$$