June 24^{th}, 2021

$$a_n = O(b_n)$$

$\exists\, c > 0$

$$0 < \frac{a_n}{b_n} \leq c$$

$$a_n = o(b_n)$$

$\forall\, \varepsilon > 0 \quad \exists\, N > 0$ st

$$0 < \frac{a_n}{b_n} < \varepsilon \quad \forall\, n \geq N$$

$\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ — two sequences (non-negative)

"Big O" $a_n = O(b_n)$ $\qquad$ $O \equiv$ order

if $\exists\ c > 0$ such that $0 \leq a_n \leq c\, b_n \quad \forall n \geq 1$

if $b_n \neq 0$

$\forall n \geq 1 \quad 0 \leq \dfrac{a_n}{b_n} \leq C$ $\qquad$ (Sequence of Ratios is bounded)

"little o" $a_n = o(b_n)$

$\forall\ \varepsilon > 0 \quad \exists\ N \geq 1$ s.t $0 \leq a_n \leq \varepsilon\, b_n \quad \forall n \geq N$

if $b_n \neq 0$

$\forall\ \varepsilon > 0 \quad \exists\ N \geq 1$ s.t $0 \leq \dfrac{a_n}{b_n} \leq \varepsilon \quad \forall n \geq N$

$\left(\dfrac{a_n}{b_n} \longrightarrow 0 \text{ as } n \to \infty \right)$

Sequence of Ratios Converges to 0.

1. Let $a, b : \mathbb{N} \to \mathbb{R}_+$ be two sequences
   - $a_n = O(b_n)$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $a_n \leq cb_n$ for all $n \geq N_0$
   - $a_n = o(b_n)$ if for every $\epsilon > 0$ there exists N_0 such that $a_n \leq \epsilon b_n$ for all $n \geq N_0$
2. Consider the following sets of sequences:

   (a) $a_n = n^3 + 5n^2 + 15$ and $b_n = n^3 + 7n + 8$

   (b) $a_n = n^3 + 5n^2 + 15$ and $b_n = n^3 + 700000n + 1000$

   (c) $a_n = n^3 + 5n^2 + 15$ and $b_n = 0.0005n^4 + 7n^2 + 8$

   (d) $a_n = n^3 + 5n^2 + 15$ and $b_n = 2^n$

   i. The below R-code is available in Dropbox shared folder BIGO.R.

```
> #writing sequences as functions
> a <- function(n){ n^3+ 5*n^2+ 15}
> b1 <- function(n){n^3+ 7*n+ 8}
> b2 <- function(n){n^3+ 700000*n+ 1000}
> b3 <- function(n){0.0005*n^4+ 7*n^2+ 8}
> b4 <- function(n){2^n }
> # setting the number of steps
> n = seq(1, 100, by = 1)
> #calculating the ratio of sequences
> c1  <-  a(n)/b1(n)
> c2  <-  a(n)/b2(n)
> c3  <-  a(n)/b3(n)
> c4  <-  a(n)/b4(n)
> #setting chart in plot to four spaces
> par(mfrow= c(2,2))
> #plotting the sequences in one chart
> plot(c1~n, cex= 0.2, col= "#d55e00", xlab= "n", ylab= "a/b1", main = "(a)")
> plot(c2~n, cex= 0.2, col= "#cc79a7", xlab= "n", ylab= "a/b2", main = "(b)")
> plot(c3~n, cex= 0.2, col= "#0072b2", xlab= "n", ylab= "a/b3", main = "(c)")
> plot(c4~n, cex= 0.2, col= "#009e73", xlab= "n", ylab= "a/b4", main = "(d)")
>
```

   In R-studio cloud or elsewhere please run the above code to obtain the plots.

   ii. From the plots can you guess for each of (a), (b), (c), (d), if $a_n = O(b_n)$ or $a_n = o(b_n)$.

   iii. Change the R and plot till $n = 5000$ for the sequences mentioned in (a), (b), (c), and (d).

   iv. Decide (with proof) whether $a_n = O(b_n)$ or $a_n = o(b_n)$ for sequences in (a), (b), (c), (d).
3. For each of the following indicate whether $a_n = O(b_n)$, or $a_n = o(b_n)$

   (a) $a_n = n^3 + 5n^2 + 15$ and $b_n = n^3 + 7n + 8$

   (b) $a_n = nb^n$, for $b \in (0, 1)$ and $b_n = \frac{1}{n^4}$

- In (i), (ii) (iii) – plotted $\frac{a_n}{b_n}$ and guessed whether $a_n = \Theta(b_n)$ or $a_n = o(b_n)$

(a) $a_n = n^3 + 5n^2 + 15$
$b_n = n^3 + 7n + 8$

$n \geq 1 \Rightarrow a_n \geq 21 \Rightarrow a_n \geq 0 \quad \forall n \geq 1$ — (*)
$b_n \geq 16 \Rightarrow b_n > 0 \quad \forall n \geq 1$

Consider $\frac{a_n}{b_n} = \frac{n^3 + 5n^2 + 15}{n^3 + 7n + 8}$

– Divide numerator and denominator by n^3

$$\frac{a_n}{b_n} = \frac{1 + \frac{5}{n} + \frac{15}{n^3}}{1 + 7/n^2 + 8/n^3} \quad \text{— (++)}$$

– lower bound:

$(*) \Rightarrow 0 \leq \frac{a_n}{b_n}$

$\alpha \geq 0$
$a \leq b$
$\Rightarrow \alpha a \leq \alpha b$

– upper bound

(+) $n \geq 1 \Rightarrow \frac{1}{n} \leq 1, \quad \frac{1}{n^2} \leq 1, \quad \frac{1}{n^3} \leq 1$

$\Rightarrow 1 + 5/n + 15/n^3 \leq 1 + 5(1) + 15(1) \leq 1 + 5 + 15 = 21$

$\frac{1}{n} \geq 0, \quad \frac{1}{n^2} \geq 0, \quad \frac{1}{n^2} \geq 0$

$c \in \mathbb{R}$
$a \leq b$
$\Rightarrow a + c \leq b + c$

$\Rightarrow 1 + 7/n^2 + 8/n^3 \geq 1 + 0 + 0 = 1$

$$\boxed{\begin{array}{l} x \geq 1 \\ \Rightarrow \frac{1}{x} \leq 1 \end{array}} \Leftarrow \frac{1}{1+\frac{7}{n^2}+\frac{8}{n^3}} \leq 1$$

$$\frac{a_n}{b_n} = \frac{1+\frac{5}{n}+\frac{15}{n^3}}{1+\frac{7}{n^2}+\frac{8}{n^3}} \leq 21.1$$

(c) $a_n = n^3 + 5n^2 + 15$

$b_n = 0.0005n^4 + 7n^2 + 8$

Guess: $a_n = o(b_n)$

Let $\alpha = 0.0005 \Rightarrow b_n = \alpha n^4 + 7n^2 + 8$

$n \geq 1$ $\quad a_n \geq 1+5+15 = 21 \geq 0$

$\quad b_n \geq \alpha + 7 + 8 \geq 15 > 0$ } (*)

$$\therefore \quad \frac{a_n}{b_n} = \frac{n^3 + 7n^2 + 15}{\alpha n^4 + 7n^2 + 8}$$

<u>Lower bound</u>

From (*) $\quad 0 \leq \frac{a_n}{b_n}$

Let $\varepsilon > 0$ be given

<u>ToDo</u> :- need to find $N \geq 1$ st

$$\frac{a_n}{b_n} \leq \varepsilon \quad \forall n \geq N$$

$$\frac{a_n}{b_n} = \frac{n^3+7n^2+15}{\alpha n^4+7n^2+8}$$

(Divide by n^4)

$$\frac{a_n}{b_n} = \frac{\frac{1}{n} + 7/n^2 + 15/n^4}{\alpha + \frac{7}{n^2} + \frac{8}{n^4}}$$

- $\forall n \geq 1$

$$\alpha + \frac{7}{n^2} + \frac{8}{n^4} \geq \alpha + 0 + 0 = \alpha \quad \text{--- \#}$$

$\therefore \forall n \geq 1$

$$\frac{a_n}{b_n} \leq \frac{1}{\alpha}\left(\frac{1}{n} + \frac{7}{n^2} + \frac{15}{n^4}\right)$$

<u>SubTodo</u> :- Find $N \geq 1$ st

$$\frac{1}{n} + \frac{7}{n^2} + \frac{15}{n^4} \leq \varepsilon\alpha, \quad \forall n \geq N$$

– <u>Subtodo2</u> :-

<u>Find $N \geq 1$</u> :

$$\frac{1}{n} \leq \frac{\varepsilon\alpha}{3}$$

$$\frac{7}{n^2} \leq \frac{\varepsilon\alpha}{3} \quad \forall n \geq N$$

$$\frac{15}{n^4} \leq \frac{\varepsilon\alpha}{3}$$

<u>Find $N \geq 1$</u> s.t.

(1) $\frac{1}{n} \leq \frac{\varepsilon\alpha}{3}$ –

(2) – $\frac{1}{n^2} \leq \frac{\varepsilon\alpha}{21}$ $\quad \forall n \geq N$

(3) $\frac{1}{n^4} \leq \frac{\varepsilon\alpha}{45}$

$$N_1 \geq \frac{3}{\varepsilon\alpha} \implies \frac{1}{n} \leq \frac{\varepsilon\alpha}{3} \quad \forall\, n \geq N_1$$

$$N_2 \geq \sqrt{\frac{21}{\varepsilon\alpha}} \implies \frac{1}{n^2} \leq \frac{\varepsilon\alpha}{21} \quad \forall\, n \geq N_2$$

$$N_3 \geq \left(\frac{45}{\varepsilon\alpha}\right)^{\frac{1}{4}} \implies \frac{1}{n^4} \leq \frac{\varepsilon\alpha}{45} \quad \forall\, n \geq N_3$$

Take $N = \max(N_1, N_2, N_3)$

$$\implies \frac{1}{n} \leq \frac{\varepsilon\alpha}{3}$$

$$\frac{1}{n^2} \leq \frac{\varepsilon\alpha}{21} \quad \forall\, n \geq N$$

$$\frac{1}{n^4} \leq \frac{\varepsilon\alpha}{45}$$

$\therefore$ SubToDo 2 ✓

$\implies$ SubToDo ✓

$\implies$ ToDo ✓

As $\varepsilon > 0$ was arbitrary we have shown

$a_n = o(b_n)$