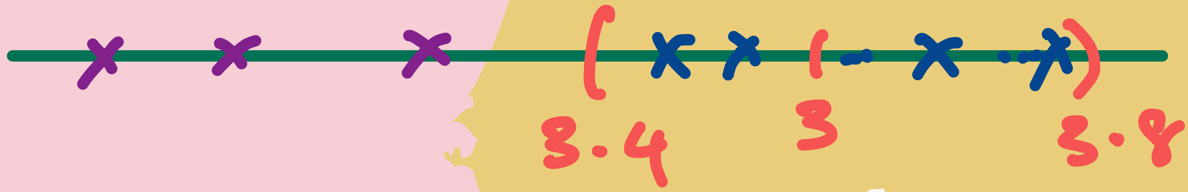
$\exists N \geq 1$:

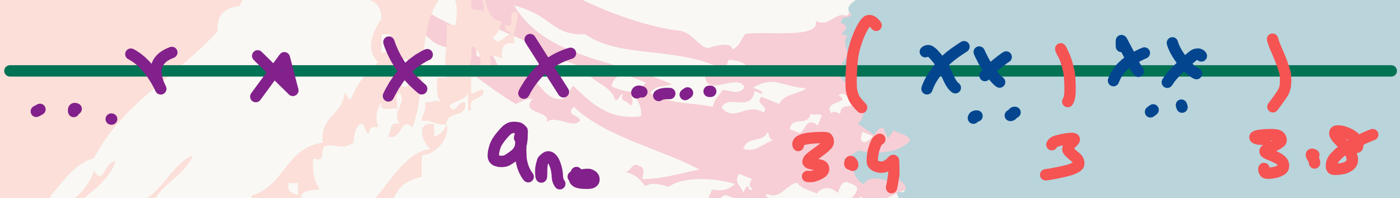


$\forall n \geq N \quad a_n \in (3.4, 3.8)$

[All but finitely many]

$\forall N \geq 1 \quad \exists n_0 > N$

$a_{n_0} \notin (3.4, 3.8)$



[Infinitely many]

June 18th, 2021

- $O(\cdot)$ & $o(\cdot)$

June 18th, 2021 Name ______________________

SWMS-Worksheet 4 in Logic

1. A sequence $\{a_n\}$ is a bounded sequence if there is a $M > 0$ such that a_n is in the interval $(-M, M)$ for all $n \in \mathbb{N}$.

   (a) Provide an example of a bounded sequence: which converges and which does not converge to a real number.

   MSPH → (b) Write a logical statement[1] that is equivalent to saying that the sequence a_n is bounded.

   SYAA → (c) Write a logical statement that is equivalent to saying that the sequence a_n is not bounded.

2. Find an example of a sequence that satisfies the below statements and then write the below statements using logical notation:

   NSHAU → (a) For every $\epsilon > 0$ there *are infinitely many* n such that distance of a_n to 0 is less than ϵ.

   YAASPM → (b) For every $\epsilon > 0$ *for all but finitely many* n the distance of a_n to 0 is less than ϵ.

3. Let $a, b : \mathbb{N} \to \mathbb{R}_+$ be two sequences

   - $a_n = O(b_n)$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $a_n \leq cb_n$ for all $n \geq N_0$
   - $a_n = o(b_n)$ if for every $\epsilon > 0$ there exists N_0 such that $a_n \leq \epsilon b_n$ for all $n \geq N_0$

   For each of the following indicate whether $a_n = O(b_n)$, or $a_n = o(b_n)$

   (a) $a_n = n^3 + 5n^2 + 15$ and $b_n = n^3 + 7n + 8$

   (b) $a_n = nb^n$, for $b \in (0, 1)$ and $b_n = \frac{1}{n^4}$

[1] **Logical Notation:** • $\forall$ to mean for all; • $\exists$ to mean there exists; • $\implies$ to mean implies; and • $\iff$ to mean equivalent.

1 (a) $a_n = \sin(n)$

$a_n = 1/n^2$ $\Big\}$ To show Bounded

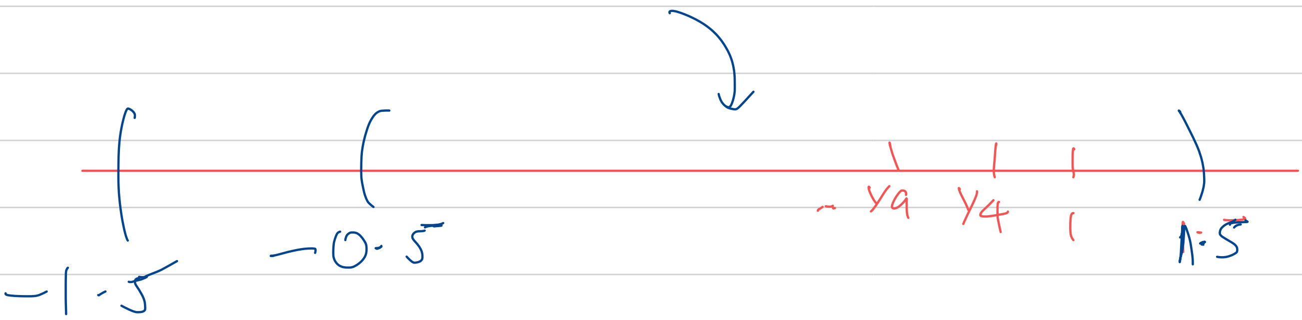
- $a_n = \sin(n)$

$M = 1.5$ $\quad -1 \leq a_n = \sin(n) \leq 1 \quad \forall n \geq 1$

$\Rightarrow \quad a_n \in (-1.5, 1.5) \quad \forall n \geq 1$

$\Rightarrow$ a_n is in the interval $(-1.5, 1.5)$ always

- $a_n = \frac{1}{n^2}$



$M = 1.5$ $\quad$ observe

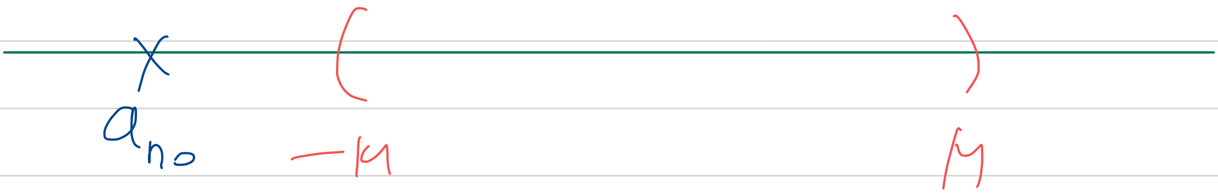
$\forall n \geq 1, \quad a_n = \frac{1}{n^2} \geq 0 > -1.5$

$\forall n \geq 1 \quad a_n = \frac{1}{n^2} \leq 1 < 1.5$

$\therefore$ a_n is in the interval $(-1.5, 1.5)$

1 (b) $\exists M > 0$ st $-M < a_n < M, \forall n \geq 1$

(c) $\forall M > 0 \quad \exists n_0 : \quad a_{n_0} \notin (-M, M)$



Example :- $a_n = n$

let $M > 0$. $n_0 = M+1 \implies a_{n_0} \notin (-M, M)$

---

2 (a) $\forall \varepsilon > 0$ there are infinitely many a_n that belong to $(-\varepsilon, \varepsilon)$

- $a_n = \frac{1}{n}$

- $a_n = \frac{1}{10^n}$

$a_n = n^2$ (X)

$a_n = n$

let $\varepsilon > 0$

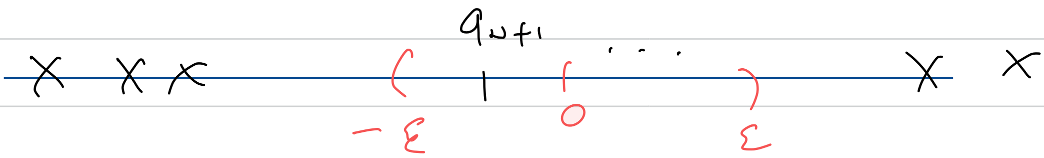
$a_n = 0$ $\quad |a_n - 0| = 0 < \varepsilon \quad \forall n \geq 1$

$\implies$ Every one is within ε of 0

$\implies$ Infinitely many $\{a_n\}_{n \geq 1}$ within distance ε of 0.

2 (b) $\forall \varepsilon>0$, All but finitely many $\{a_n\}_{n\geq 1}$ belong to $(a-\varepsilon, a+\varepsilon)$

$\forall \varepsilon>0 \quad \exists N \geq 1$



let $\varepsilon>0$ be given

As $a_n = 0 \;\; \forall n\geq 1 \implies |a_n - 0| = 0 < \varepsilon \;\; \forall n \geq 1$

(a) "infinitely many"		(b) "all but finitely many"
$a_n = 0$	$\Longleftarrow$ ✓	$a_n = 0$.
$a_n = \frac{1}{n}$		$a_n = \frac{1}{n}$.
$a_n = \frac{1}{10n}$		$a_n = \frac{1}{10n}$
$a_n = \begin{cases} -1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$	$\implies$	(✗).

let $\varepsilon>0$ be given

$$|a_n - 0| = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

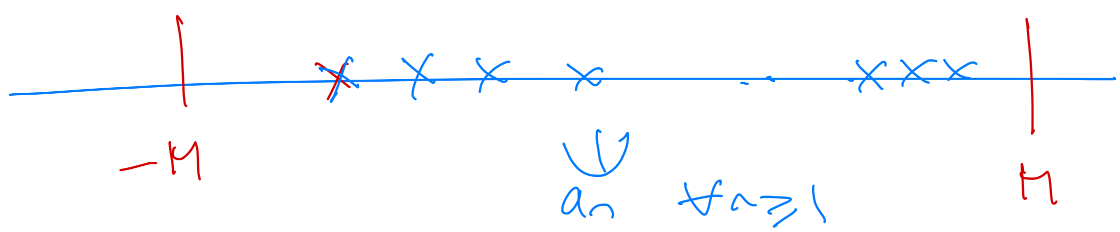
$\forall N \geq 1$, let $m = 2N$ $\quad |a_m - 0| = 0 < \varepsilon$

$\therefore \; a_m \in (-\varepsilon, \varepsilon)$

1. A sequence $\{a_n\}$ is a bounded sequence if there is a $M > 0$ such that a_n is in the interval $(-M, M)$ for all $n \in \mathbb{N}$.
   (a) Provide an example of a bounded sequence.
   R P US (b) Write a logical statement[1] that is equivalent to saying that the sequence a_n is bounded.
   P A A S (c) Write a logical statement that is equivalent to saying that the sequence a_n is not bounded.
2. Find an example of a sequence that satisfies the below statements and then write the below statements using logical notation:
   RS AP (a) For every $\epsilon > 0$ there *are infinitely many* n such that distance of a_n to 3 is less than ϵ.
   A PU M (b) For every $\epsilon > 0$ *for all but finitely many* n the distance of a_n to 3 is less than ϵ.
3. Let $a, b : \mathbb{N} \to \mathbb{R}_+$ be two sequences
   - $a_n = O(b_n)$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $a_n \le c b_n$ for all $n \ge N_0$
   - $a_n = o(b_n)$ if for every $\epsilon > 0$ there exists N_0 such that $a_n \le \epsilon b_n$ for all $n \ge N_0$

   For each of the following indicate whether $a_n = O(b_n)$, or $a_n = o(b_n)$
   (a) $a_n = n^3 + 2n^2 + 10$ and $b_n = n^3 + 6n + 1$
   (b) $a_n = nb^n$, for $b \in (0, 1)$ and $b_n = \frac{1}{n^6}$

Bounded sequence by picture



[1]**Logical Notation:** • $\forall$ to mean for all; • $\exists$ to mean there exists; • $\implies$ to mean implies; and • $\iff$ to mean equivalent.

1 ⓐ

$$a_n = (-1)^n$$

$$a_n = \frac{1}{n}$$

$$a_n = 5$$

To show sequence is bounded,

- $a_n = (-1)^n$

$M = 2$

$$a_n = \begin{cases} 1 & n - \text{even} \\ -1 & n - \text{odd} \end{cases}$$

$$\Rightarrow \quad -2 < a_n < 2 \quad \forall \, n \geq 1$$

$\Rightarrow$ a_n is in the interval $(-2, 2)$ for all $n \in \mathbb{N}$

- $a_n = \frac{1}{n}$

$M = 2$

other choices

3
4
⋮

1.5
1.4
1.3
⋮

$$a_n = \frac{1}{n} \quad \forall n \geq 1$$

As $0 \leq \frac{1}{n} \leq 1 \quad \forall n \geq 1$

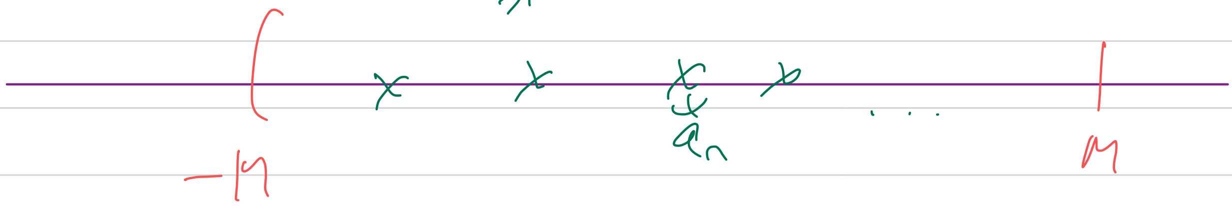
$$\Rightarrow \quad -2 < a_n < 2 \quad \forall n \geq 1$$

$\Rightarrow$ a_n is in the interval $(-2, 2)$ for all $n \in \mathbb{N}$

(b) $\exists M > 0 \quad \forall n \geq 1 \quad a_n \in (-M, M)$

$$\Updownarrow$$

$\{a_n\}_{n \geq 1}$ is bounded



(c) $\forall M > 0 \quad \exists n_0 \geq 1 \quad a_{n_0} \in (-M, M)$

$$\Updownarrow$$

$\{a_n\}_{n \geq 1}$ is not bounded

Example :- $a_n = n$

let $M > 0$ be given

$n_0 = M + 1$

$M + 1 = a_{n_0} \notin (-M, M)$

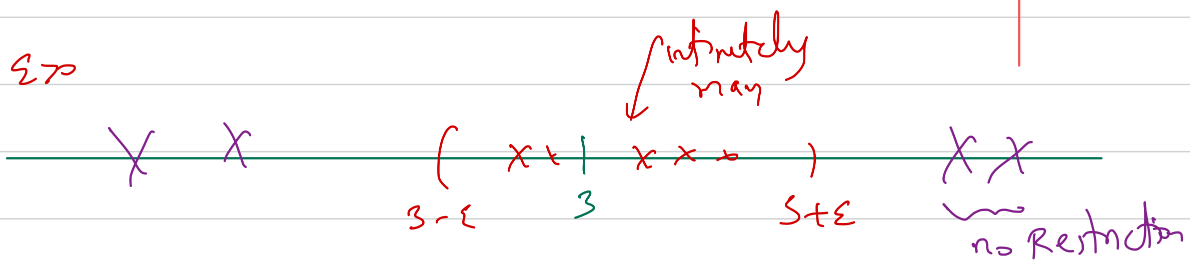
2 (a) $a_n = 3 \quad \forall n \geq 1$

$a_n = 3(-1)^n \quad \forall n \geq 1$

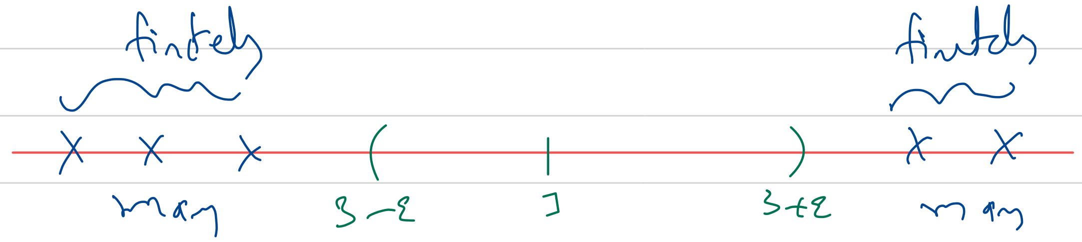
2 (b) $a_n = \begin{cases} 1 & n=1 \\ 2 & n=2 \\ 3 & n \geq 3 \end{cases}$

(a) infinitely many		(b) all but finitely
$a_n = 3$	$\Longrightarrow$	$a_n = 3$
$a_n = 3(-1)^n$		✗
	$\Longleftarrow$	

(a) $\forall \varepsilon > 0$



(b)



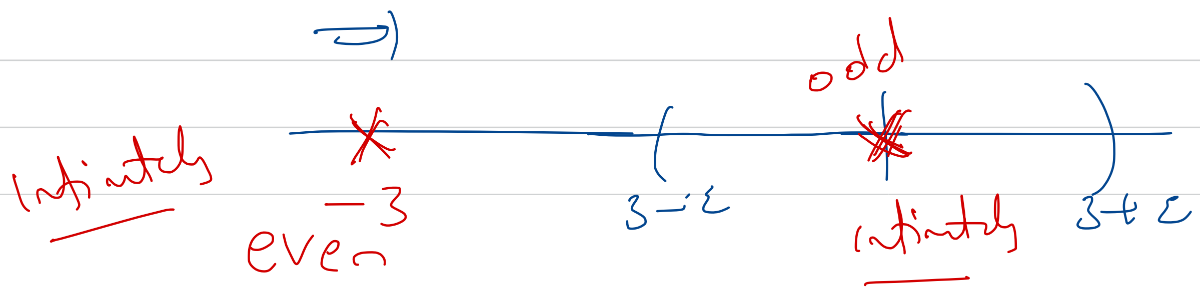
$a_n = 3(-1)^n$

let $\varepsilon > 0$ be given.

$$|a_n - 3| = \begin{cases} 0 & \text{if } n \text{ is even} \\ 6 & \text{if } n \text{ is odd} \end{cases}$$

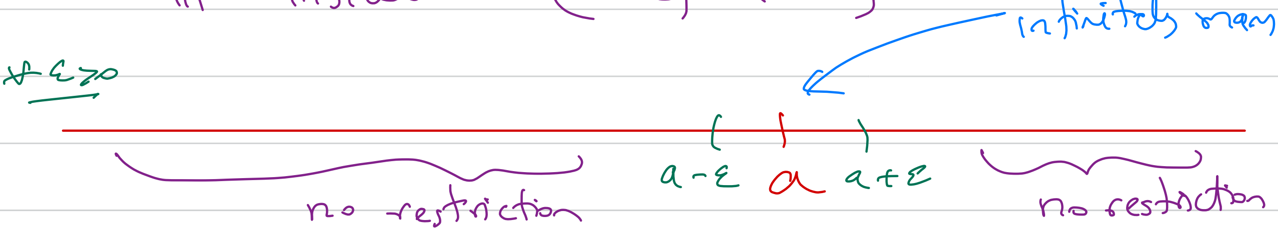
$\forall N \geq 1 \qquad m = 2N+1 \qquad a_m \in (-3+\varepsilon, 3+\varepsilon)$

$\Longrightarrow$



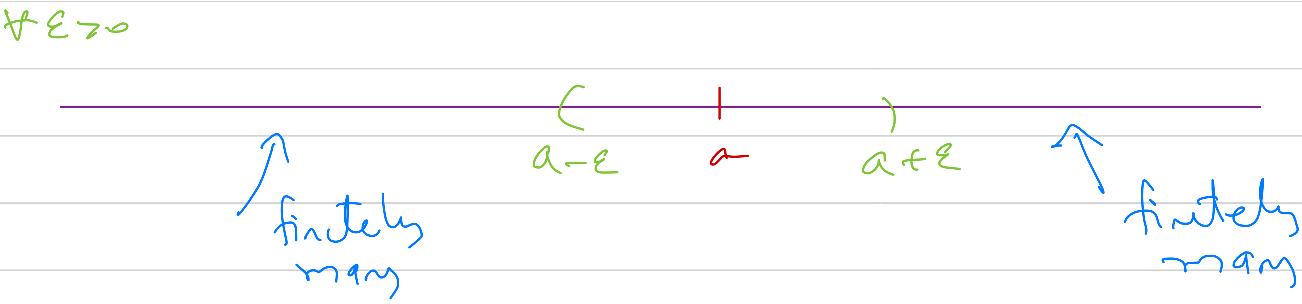
Fix $a \in \mathbb{R}$ and $\{a_n\}_{n\geq 1}$

(a) for all $\varepsilon > 0$ there are infinitely many a_n inside $(a-\varepsilon, a+\varepsilon)$



$$\forall \varepsilon > 0 \quad \forall N \geq 1 \quad \exists n_0 > N \quad a_{n_0} \in (a-\varepsilon, a+\varepsilon)$$

(b) for all $\varepsilon > 0$ all but finitely many a_n are inside $(a-\varepsilon, a+\varepsilon)$

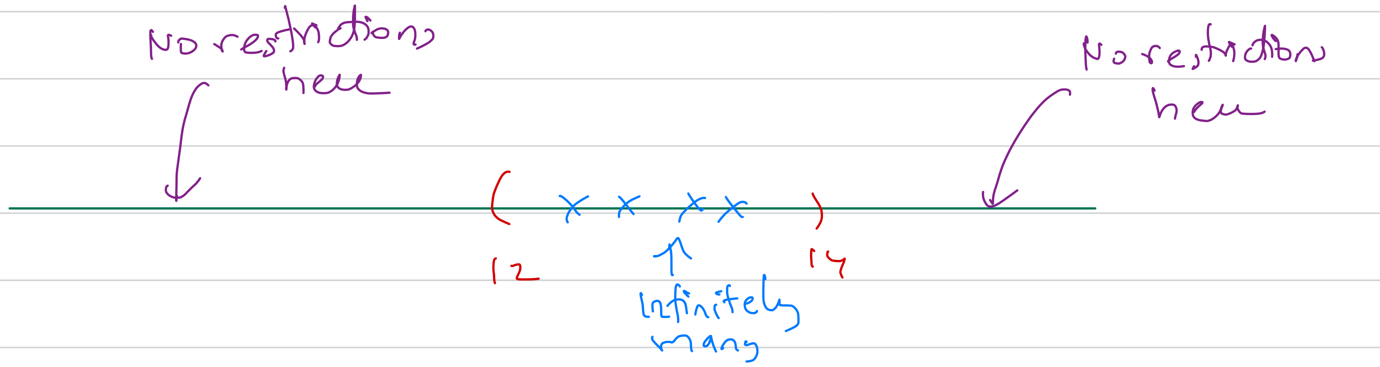


$$\forall \varepsilon > 0 \quad \exists N \geq 1 \text{ st } \forall n \geq N \quad a_n \in (a-\varepsilon, a+\varepsilon)$$

i.e. $\exists N \geq 1$ such that only $\{a_1, \ldots, a_N\}$ may not belong to $(a-\varepsilon, a+\varepsilon)$

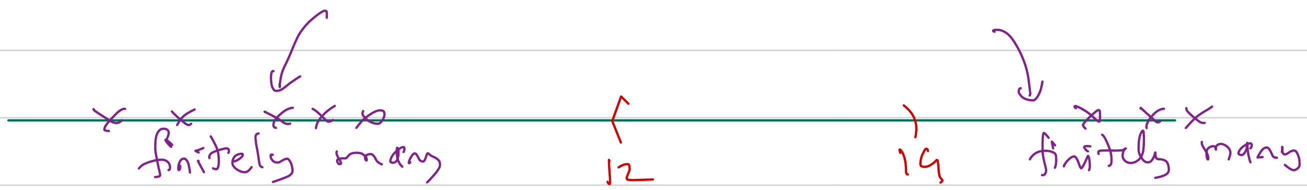
$\{a_n\}_{n\geq 1}$

2 (a) • infinitely many elements are inside $(12, 14)$



• $\forall N \geq 1$, $\exists n_0 > N$ $a_{n_0} \in (12, 14)$

2 (b) all but finitely many elements are inside $(12, 14)$.



• $\exists N > 0$ such that

$\forall n \geq N$ $a_n \in (12, 14)$

$\exists N > 0$ s.t. $\{a_1, \dots, a_N\}$ may not satisfy property of being inside $(12, 14)$