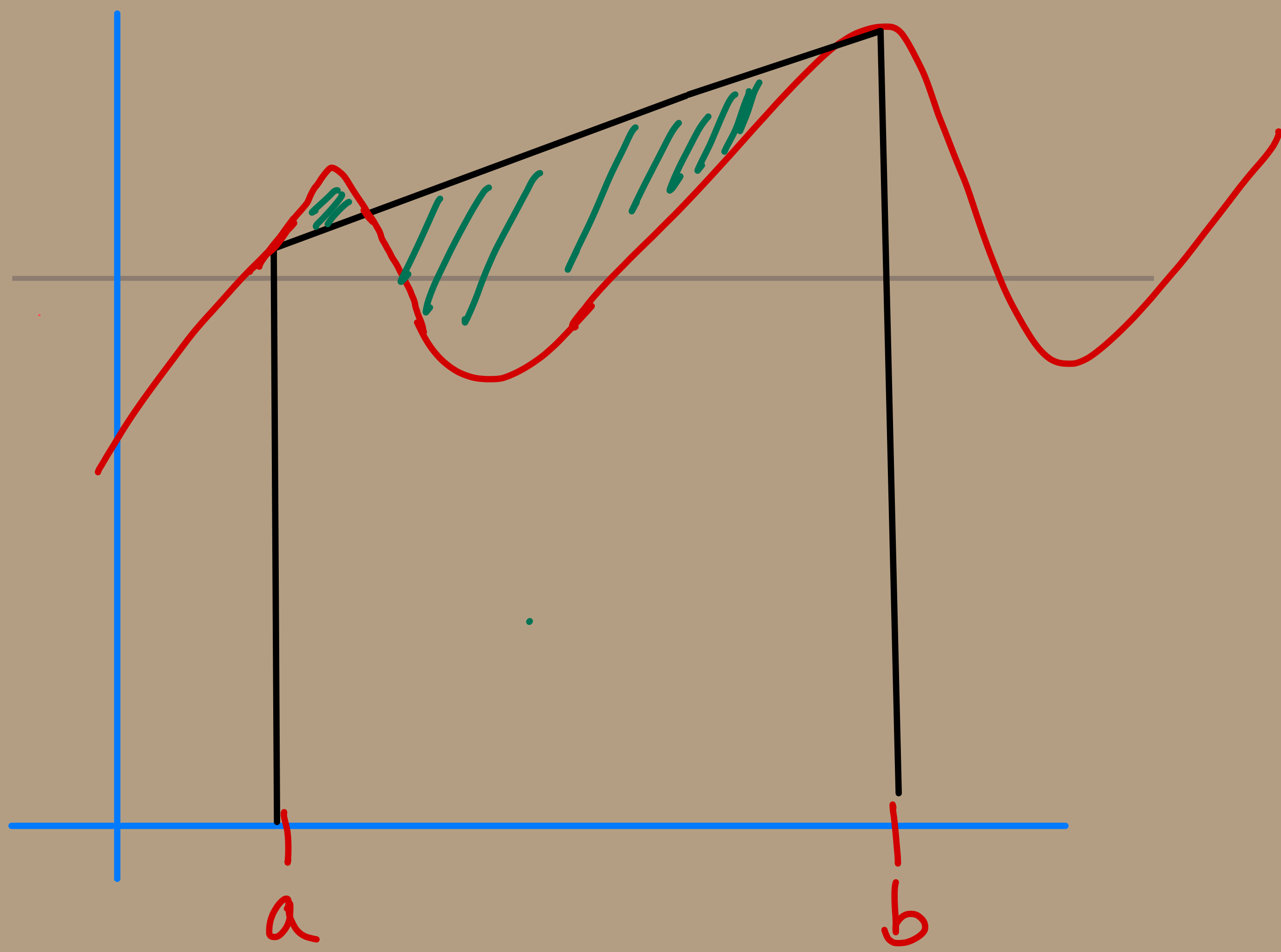
June 15th 2021

Integral Approximation : Trapezoid



and Simpson Rule

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{6} f'''(\xi)$$

Worksheet 2 :

$$p_2(x) = \frac{(x-a)(x-b)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} f\left(\left(\frac{a+b}{2}\right)\right) + \frac{\left(x-\frac{a+b}{2}\right)(x-b)}{\left(a-\frac{a+b}{2}\right)(a-b)} f(a)$$

$$+ \frac{\left(x-\frac{a+b}{2}\right)(x-a)}{\left(b-\left(\frac{a+b}{2}\right)\right)(b-a)} f(b)$$

$$\int_a^b p_2(x)\,dx = \frac{f\left(\left(\frac{a+b}{2}\right)\right)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \int_a^b (x-a)(x-b)\,dx$$

$$+ \frac{f(a)}{\left(a-\frac{a+b}{2}\right)(a-b)} \int_a^b \left(x-\frac{a+b}{2}\right)(x-b)\,dx$$

$$+ \frac{f(b)}{\left(b-\left(\frac{a+b}{2}\right)\right)(b-a)} \int_a^b \left(x-\frac{a+b}{2}\right)(x-a)\,dx$$

Use : $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$; $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$

Ex

Ans

$$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Simpson's Rule (Quadratic)

$$\int_a^b f(x)\,dx \approx \int_a^b p_2(x)\,dx$$

$p_2(\cdot)$ is a quadratic passing through $(a, f(a))$, $(b, f(b))$ and $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_0^1 x^3\,dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

$$\begin{bmatrix}\text{Simpson's} \\ \text{Rule}\end{bmatrix} \approx \frac{1-0}{6}\left[0^3 + 4\left(\frac{1}{2}\right)^3 + 1^3\right]$$

$$= \frac{1}{6}\left[\frac{1}{2} + 1\right] = \frac{1}{4}$$

It is also true $\forall\, a, b \in \mathbb{R}$ $a < b$

$$\int_a^b x^3\,dx \overset{\text{Simpson}}{\underset{\text{Rule}}{\simeq}} \frac{b^4}{4} - \frac{a^4}{4}$$

Observation: Simpson's rule is exact for all $f: [a,b] \to \mathbb{R}$ where f is a cubic polynomial.

Questions :

- Riemann Sums
- Trapezoid Rule
- Simpsons Rule

} Approximation

Can we quantify the errors?

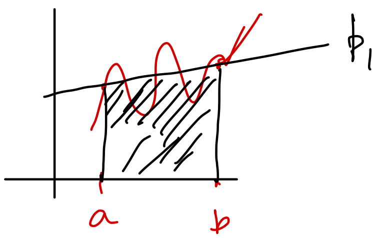
1. Suppose we are given $f : [a, b] \to \mathbb{R}$ and the values of $(a, f(a))$, $(\frac{a+b}{2}, f(\frac{a+b}{2}))$, $(b, f(b))$.

Question: How to compute area under the curve?

: values that are known

If f is a straight line then $f \equiv p_1$

(a) Find the line $p_1 : [a, b] \to \mathbb{R}$ passing through $(a, f(a))$ and $(b, f(b))$



$$\equiv \quad p_1(x) = \frac{x-a}{b-a} f(b) + \frac{x-b}{a-b} f(a)$$

(b) Find the quadratic $p_2 : [a, b] \to \mathbb{R}$ passing through $(a, f(a))$, $(\frac{a+b}{2}, f(\frac{a+b}{2}))$, $(b, f(b))$

$$p_2(x) = \frac{(x-a)(x-b)}{(\frac{a+b}{2}-a)(\frac{a+b}{2}-b)} f\left(\frac{a+b}{2}\right) + \frac{(x-\frac{a+b}{2})(x-a)}{(b-\frac{a+b}{2})(b-a)} f(b) + \frac{(x-\frac{a+b}{2})(x-b)}{(a-\frac{a+b}{2})(a-b)} f(a)$$

(c) Fill in the following table:

	Approximation of $\int_a^b f(x)dx$
(Linear) $\int_a^b p_1(x)dx$	$\frac{b-a}{2}[f(b) + f(a)]$
(Quadratic) $\int_a^b p_2(x)dx$	$\frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$

when are these approximation? exact

Suppose f is a quadratic & we have

- $(a, f(a))$, $(b, f(b))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$

- there is a unique quadratic passing through them.

$\Rightarrow \boxed{P_2 \equiv f}$

$$\Rightarrow \int_a^b P_2(x)\,dx = \int_a^b f(x)\,dx.$$

Trapezoid Rule (Linear)

$f : [a,b] \to \mathbb{R}$

$$\int_a^b f(x)\,dx \cong \frac{b-a}{2}\left(f(a)+f(b)\right)$$

Exact if f is a linear function

Simpsons Rule (Quadratic)

$f : [a,b] \to \mathbb{R}$

$$\int_a^b f(x)\,dx \cong \int_a^b p_2(x)\,dx$$

where $p_2(\cdot)$ was quadratic and it passed through

- $(a, f(a))$, $(b, f(b))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Exact if f is ~~a~~ upto 3-degree polynomial ~~Quadratic~~ function

Observation :-

$$\int_a^b x^3\,dx = \frac{x^4}{4}\Big|_a^b = \frac{b^4 - a^4}{4}$$

$f(x) = x^3 \qquad f : [a,b] \to \mathbb{R}$

- Simpson's rule

$$\frac{(b-a)}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3 \right]$$

$$= \cdots$$

$$= \frac{b^4}{4} - \frac{a^4}{4}$$

$$\int_a^b x^3\,dx = \text{Simpson's rule approximation.}$$

$\square$

1. Suppose we are given the below data about $f : [0, \cancel{\ }\ \tfrac{\pi}{2}] \to \mathbb{R}$.

Table value Matches

$f(x) = \sin(x) \longrightarrow$

$f(0)$	$f(\frac{\pi}{4})$	$f(\frac{\pi}{2})$
0	$\frac{1}{\sqrt{2}}$	1

   (a) Use the part (c) to provide an approximation of $\int_0^{\frac{\pi}{4}} f(x)dx$.

   $\boxed{\text{Trapezoid} \quad \text{Use Simpson rule}}$

   (b) Suppose $f(x) = \sin(x)$ then quantify the error in each approximation.

   $|||$

   $\int_0^{\pi/2} \sin(x)dx$

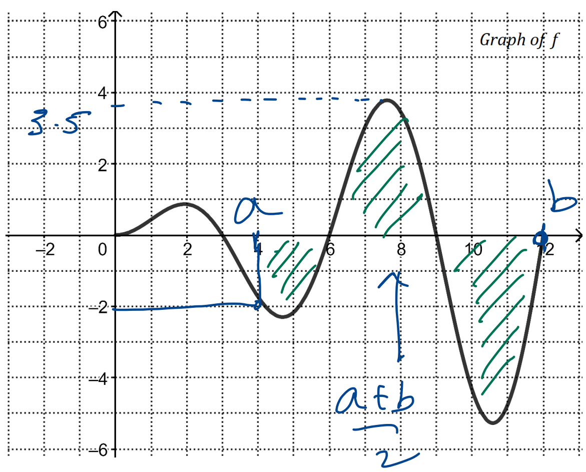
   (c) Are their functions for which the approximation(s) will be exact ?

2. The graph of a function $f(t)$ is shown. Use it to answer the following questions.

Average value

$$\frac{1}{12-4}\int_4^{12} f(x)dx$$

$$= 1/8 \int_4^{12} f(x)dx$$



   (a) Using 1(c) provide an approximation of the **average value** of this function over the interval [4, 12].

   (b) Can you provide a better approximation of the same using 1(c) ?

(a) $$T_{Rule} := \frac{\frac{\pi}{2} - 0}{2}[0 + 1] = \frac{\pi}{4} \simeq 0.785$$

$$S_{Rule} := \frac{\frac{\pi}{2} - 0}{6}\left[0 + 4\frac{1}{\sqrt{2}} + 1\right] \cong 1.0011$$

(b) $$I = \int_0^{\pi/2} \sin(x)dx = -\cos(x)\Big|_0^{\pi/2} = 1$$

$I - S_{Rule} = -0.0011$ } give different Errors.

$I - T_{Rule} = 0.215$

(c) T_{Rule} = Exact for linear functions

S_{Rule} = Exact upto cubic functions

2 (a) Average value of f in $[4,12]$

$$= \frac{1}{8}\int_4^{12} f(x)\,dx$$

Simpson's rule

$$\cong \frac{1}{8}\cdot\frac{[12-4]}{6}\left[f(4)+4f(8)+f(12)\right]$$

$$= \frac{1}{\cancel{8}}\cdot\frac{\cancel{8}}{6}\left[-2+4(3.5)+0\right]$$

$$\cong 2$$

2(b) — Think about how to improve (a)

1. Suppose we are given the below data about $f : [0, 1] \to \mathbb{R}$.

Table values matches $f(x) = \sqrt{x}$

$f(0)$	$f(\frac{1}{2})$	$f(1)$
0	$\frac{1}{\sqrt{2}}$	1

Trapezoid Rule
Simpson Rule

(a) Use the part (c) to provide an approximation of $\int_0^1 f(x)dx$.

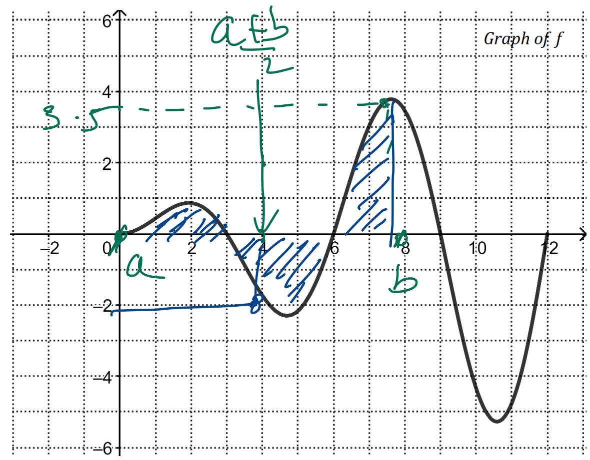
(b) Suppose $f(x) = \sqrt{x}$ then quantify the error in each approximation.

(c) Are their functions for which the approximation(s) will be exact ?

$\int_0^1 f(x)\,dx$

2. The graph of a function $f(t)$ is shown. Use it to answer the following questions.

Average value
$\frac{1}{8}\int_0^8 f(x)\,dx$



(a) Using 1(c) provide an approximation of the **average value** of this function over the interval $[0, 8]$.

(b) Can you provide a better approximation of the same using 1(c) ?

(a) $T_{Rule} = \frac{(1-0)}{2}[0+1] = \frac{1}{2}$

$S_{Rule} = \frac{(1-0)}{6}\left[0 + \frac{4}{\sqrt{2}} + 1\right] = 0.638$

(b) $I = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3} = 0.667$

$$\text{Error}\left\{\begin{array}{l} I - T_{Rule} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} = 0.166 \\ I - S_{Rule} = \frac{2}{3} - 0.638 = 0.0287 \end{array}\right.$$

(c) T_{Rule} exact for linear functions

S_{Rule} exact for f upto a cubic function.

2 (a) S_{Rule} approximation

$$= \frac{1}{8} \cdot \left[\frac{8-0}{6} \left[f(0) + 4f(4) + f(8) \right] \right]$$

$$= \frac{1}{6} \left[0 + 4(-2) + 3.5 \right]$$

$$= -\frac{4.5}{6} = -0.75$$

2(b) — Think about how to improve (c)