June 23, 2021

SWMS - DISCRETE GEOMETRY.

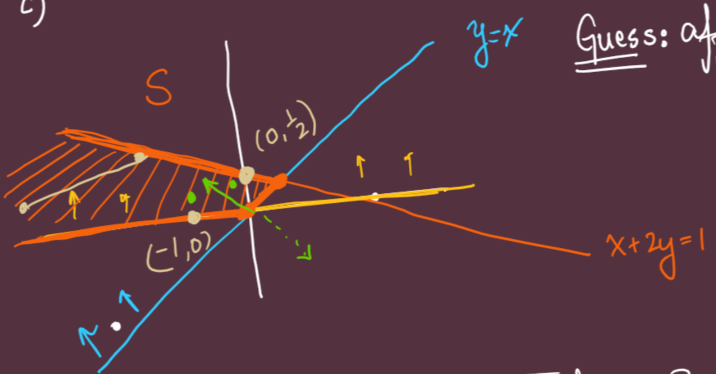
Questions to be discussed:

1. Does the difference between convex and non convex polygons have anything to do with linear/affine independence ? $\boxed{\text{Short answer: no.}}$

2. How do we visualize shapes in 4d?

$\boxed{\text{Back to Worksheet 2}}$

c)



Guess: affine hull of which vectors ?

$v \in lin(w_1, .. w_k)$
$\Rightarrow$
$-v \in lin(w_1, ..., w_k)$
$\Downarrow$
S is not a linear span.

Not an affine hull : Take : $\underbrace{3}_{} \overset{S}{(-1, 0)} - \underbrace{2}_{} \overset{S}{(0, \tfrac{1}{2})}$

$= (-3, -1) \notin S$

$\Rightarrow$ S is not an affine span.

Observations : ① both $(-1, 0)$ & $(0, \frac{1}{2})$ are on the boundary.

② unbounded. $\Rightarrow$ ① S is not a convex hull of finitely many points.

$*$

② Intuition : this is "convex."

$\rightsquigarrow$ cvx (infintely many points).

Last time : convexity :

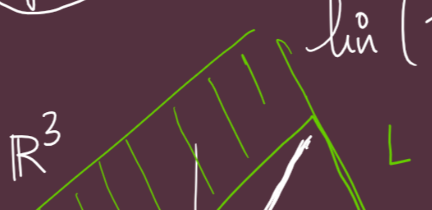
if $x, y \in S$ then $cvx(x, y) \subseteq S$.

Q. Let L be a linear subspace of $\mathbb{R}^n$. Let $v, w, u \in L$. What can you say about

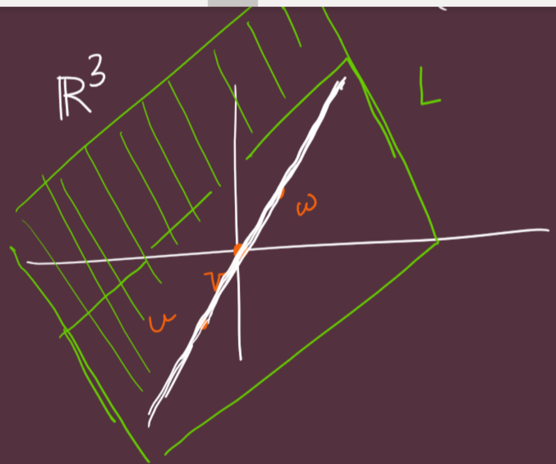
$lin(v, w, u) \subset L$

Needs proof

Linear spans are closed under taking linear combinations.



Q. Let S is an affine hull in $\mathbb{R}^n$.



Q. Let S is an affine hull in $\mathbb{R}^n$. Then the affine combination of <u>any set</u> $w_1, \dots, w_k \in S$ is always inside S.

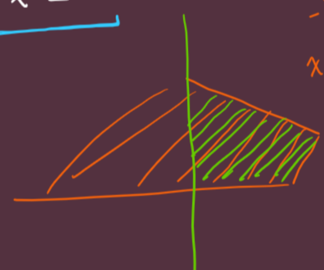
Affine hulls are closed under taking affine combination.

<u>Definition</u>. Let $v_1, \dots, v_k \in \mathbb{R}^n$. Consider the set

$$\text{cone}(v_1, \dots, v_k) = \left\{ t_1 v_1 + \dots + t_k v_k : \begin{array}{l} t_1, \dots, t_k \in \mathbb{R} \\ t_1, \dots, t_k \geq 0 \end{array} \right\}.$$

This set is called the <u>polyhedral cone</u> of $v_1, \dots, v_k$ in $\mathbb{R}^n$.

* The region S is part c) above is a polyhedral cone.
* Polyhedral cones appear as solutions of $A\vec{x} \leq b$.



$x - y \leq 0$
$-y \leq 0$
$x + 2y \leq 1$
$+$
$x \geq -2$

Part d) also gives a polyhedral cone

Solution spaces to (systems of linear equations) are affine spans $\parallel$ $(A\vec{x} = b)$

c) $x - y \leq 0, \quad y \geq 0, \quad x + 2y \leq 1$

$-y \leq 0$

$$\begin{pmatrix} x - y \\ -y \\ x + 2y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A\vec{x} \le b$$

Row reduction : $R_3 \mapsto R_3 - R_1$

does not go together!!

$$\left[\, x - y \le 0 \right.$$
$$\left[\, x + 2y \le 1 \right.$$

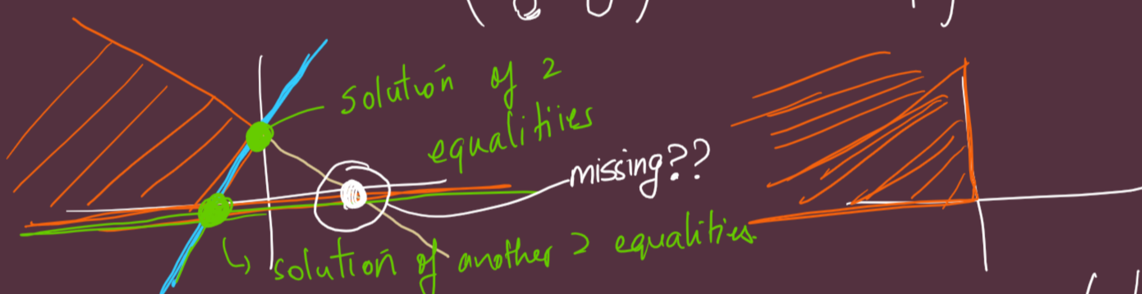
$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$R_1 \mapsto R_1 - R_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$R_3 \mapsto R_3 + 3R_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \left\{ \begin{matrix} x \le 0 \\ -y \le 0 \end{matrix} \right\}$$



WARNING: ① Gaussian elimination cannot be used to solve $Ax \le b$!

There is a method called:

② Fourier-Motzkin elimination can be used to say whether or not $\boxed{A\vec{x} \le b}$ has a solution!

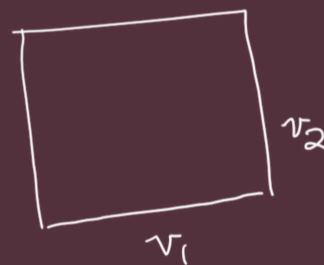
③ Even if A is invertible, $Ax \le b \not\Leftrightarrow x \le A^{-1}b$

why $\leq$?

$$-Ax \geq b$$

Linear programming

Find $v = (v_1, v_2) \in \mathbb{R}^2$ that



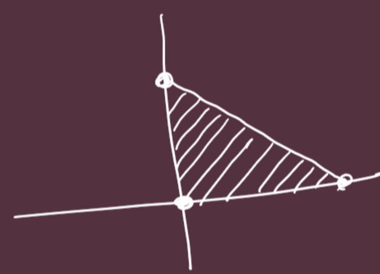
① minimize ~~maximize~~ $f(v) = -v_1 v_2$

not a linear combination of v_1 & v_2!

subject to $2v_1 + 2v_2 \leq 2$

$v_1 \geq 0,\ v_2 \geq 0.$

Constraint region



Not a linear program!

②

food types	Calories (cal/kg)	I (mg/kg)	II	III
A	c_1	a_1	a_2	a_3
B	c_2	b_1	b_2	b_3

Let x and y be the amount of type A & B, resp, being fed to the cow.

minimize $f(x,y)$: $c_1 x + c_2 y$ objective function

Subject to :

$$\left.\begin{aligned} a_1 x + b_1 y &\geq m_1 \\ a_2 x + b_2 y &\geq m_2 \\ a_3 x + b_3 y &\geq m_3 \\ x \geq 0,\ y &\geq 0. \end{aligned}\right\}$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ 0 \\ 0 \end{pmatrix}$$

This is a linear program.

$$Av \geq b$$

dummy variable

$x - y + w = 2$, $w \geq 0$

minimize $x+y$

subject to $x - y \le 2$

$2x+3y \le 5$

$x \geq 0, \ y \geq 0$

dummy variable

$x - y + w = 2, \quad w \geq 0$

$$\left\{ \begin{array}{l} \boxed{2x+3y+z=5} \\ z \geq 0 \\ x \geq 0, \ y \geq 0 \end{array} \right.$$

$Ax = b$