June 23, 2021

SWMS – DISCRETE GEOMETRY.

Questions to be discussed?

1. General algorithm to solve $A\vec{x} \le b$? (Worksheet II, parts c) & d))
2. What is the convex hull of a general set? (Worksheet III, part e)).

Part c) of Worksheet II

$x - y \le 0, \quad y \ge 0, \quad x + 2y \le 1$

$-y \le 0$

$$\begin{pmatrix} x - y \\ -y \\ x + 2y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Gaussian elimination

$\boxed{R_3 \mapsto R_3 - R_1}$ — subtracting two inequalities is a complicated business

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{matrix} x \le 0 \\ y \le 0 \end{matrix} \not\Rightarrow x - y \le 0.$$

$R_1 \mapsto R_1 - R_2$

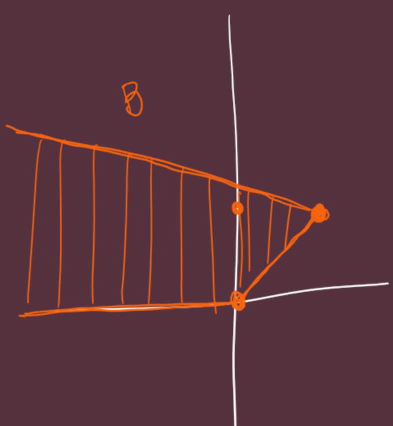
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$R_3 \mapsto R_3 + 3R_2$

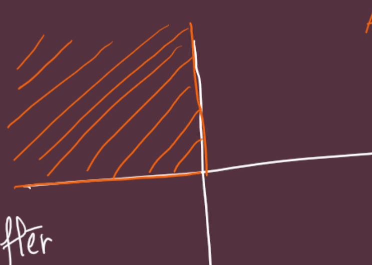
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \begin{cases} x \le 0 \\ -y \le 0 \end{cases}$$

Neither $A \supseteq B$ nor $B \supseteq A$.

B



A



After Gaussian Elimination.

- Losing some conditions
- $AI = A$ but $\boxed{AI \leq A}$ → does not current make sense for us.

$x, y \in \mathbb{R}$

$x = y \Rightarrow x \leq y.$

$x > y \Rightarrow x \neq y.$

$$\boxed{A_{n\times n}\, \vec{x}_{n\times 1} \leq b_{n\times 1}}$$

Warning: ① Gaussian Elimination cannot help with solving $Ax \leq b$.

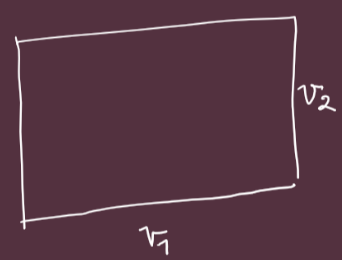
② There is a general method known as the Fourier-Motzkin elimination method to determine whether or not $A\vec{x} \leq b$ has a solution.

Q. Why care about $A\vec{x} \leq b$?

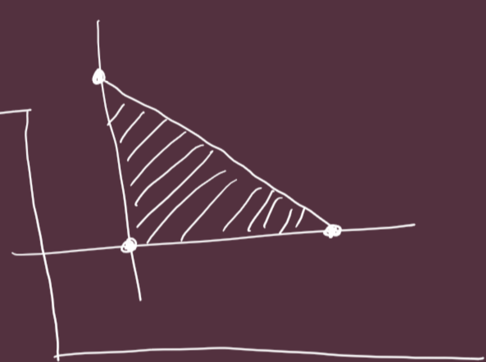
Examples: 1. Find $v = (v_1, v_2)$ that

maximizes $f(v) = v_1 v_2$ — not of the form $c_1 v_1 + c_2 v_2$!

subject to $2v_1 + 2v_2 \leq 2,\ v_1 \geq 0,\ v_2 \geq 0.$



NOT A LINEAR PROGRAM.



2.

types	cal/kg Calories	mgs/kg I	mgs/kg II	mgs/kg III
A	c_1	a_1	a_2	a_3
B	c_2	b_1	b_2	b_3

Let x and y denote the quantities (in kg) of types A & B to be fed to the cow.

To minimize:

$$f((x,y)) = c_1 x + c_2 y$$

Subject to

Subject to

$$\left.\begin{aligned} a_1x+b_1y &\geq m_1 \\ a_2x+b_2y &\geq m_2 \\ a_3x+b_3y &\geq m_3 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}\right\} \rightsquigarrow \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ 0 \\ 0 \end{pmatrix}.$$

Return to worksheet 2 Part e).

The circle $\{x^2+y^2 \leq 1\}$ is not the convex hull of a finite number of vectors in $\mathbb{R}^2$.

Definition. Let $S \subseteq \mathbb{R}^n$. Let convex hull of S is the set

$$\mathrm{cvx}(S) = \left\{ t_1v_1+\dots+t_kv_k : \begin{array}{l} k \in \mathbb{N},\ v_1,\dots,v_k \in S, \\ t_1,\dots,t_k \in \mathbb{R} \\ \sum_{j=1}^{k} t_j = 1 \text{ and } t_1,\dots,t_k \geq 0 \end{array} \right\}.$$

Earlier, $S = \{w_1,\dots,w_m\}$. Now, S can be anything in $\mathbb{R}^n$!

$\uparrow$

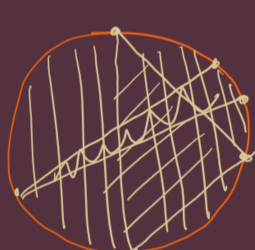
$\mathrm{cvx}(S)$ coincides with the older definitions.

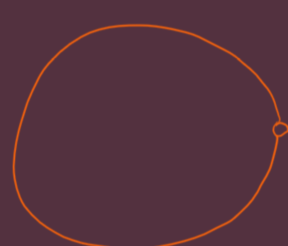
Coming back to $D = \{x^2+y^2 \leq 1\}$, we have that

Grand HW prove!

1. $D = \mathrm{cvx}\{x^2+y^2 = 1\}$
2. $D \neq \mathrm{cvx}(S)$, where $S \subsetneq \{x^2+y^2=1\}$.

$S = \{x^2+y^2=1 \ \& \ (x,y) \neq (1,0)\}$



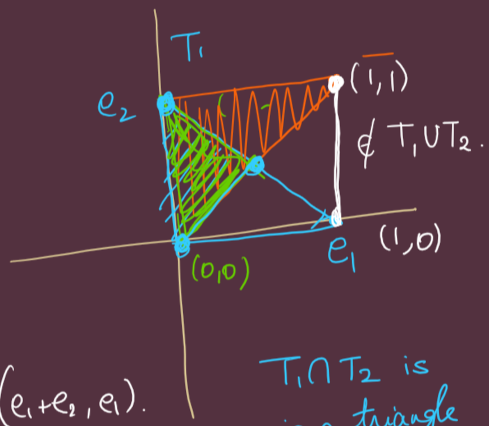


$(1,0)$ cannot be written as a convex combination of the rest of the points in D.

$\mathrm{cvx}(S) \neq D$.

of the points in D.

Worksheet 2 Part f)



$T_1 \cap T_2$ is a triangle

f) Yes, a convex hull. ✓

g)

h) No, not a convex hull. ✓
Since $T_1 \cup T_2$ does not contain ~~the~~ $\text{cvx}(e_1+e_2, e_1)$.

F) If $A = \text{cvx}(v_1, \dots, v_k)$
& $B = \text{cvx}(w_1, \dots, w_m)$, then is

$$A \cap B = \text{cvx}\{u_1, \dots, u_s\}\ ?$$

Is there a formula for u_i's in terms of v_j's and w_k's.

Q. Does $v \in A + B$?

$t_1 + \dots + t_k = 1$
$+\ s_1 + \dots + s_m = 1$ $= 2$

Find $a \in A$, $b \in B$ s.t. $\boxed{v = a + b}$

$$\left.\begin{aligned} a &= t_1 v_1 + \dots + t_k v_k \\ &+ \\ b &= s_1 w_1 + \dots + s_m w_m \end{aligned}\right\} \text{ new convex combination ?}$$

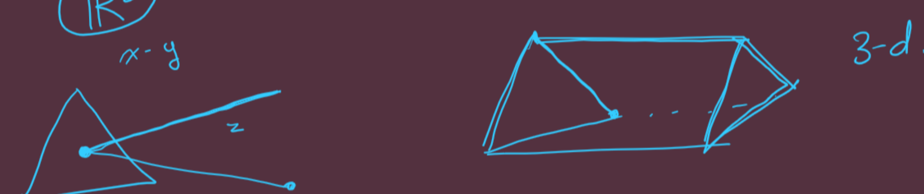
Can you write $a+b$ as

$\mathbb{R}^2$

$\mathbb{R}^2$

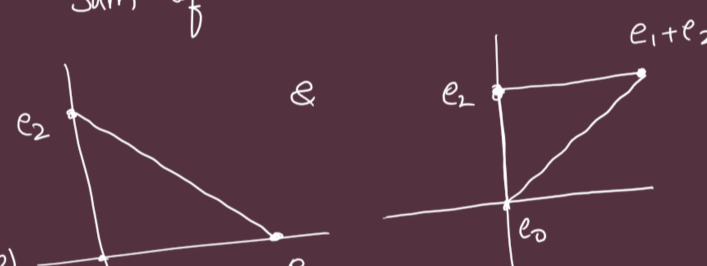
$\mathbb{R}^3$

x-y



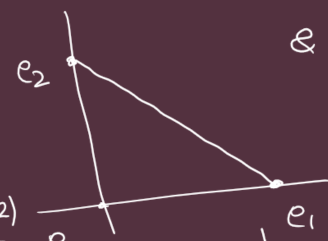
$\mathbb{R}^4$

Sum of

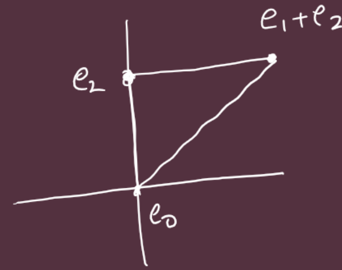


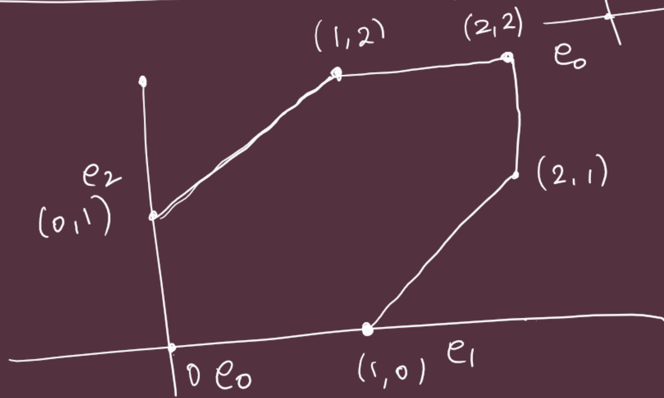
$\mathbb{R}^4$

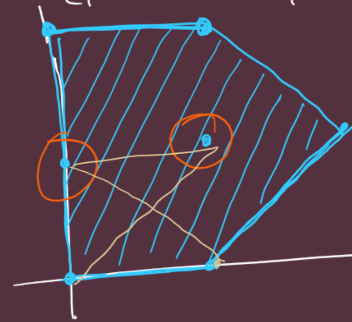
Sum of



&







claim

Strategy: only add vertices

$+$	e_0	e_1	e_2
e_0	e_0	e_1	e_2
e_1+e_2	e_1+e_2	$2e_1+e_2$	e_1+2e_2
e_2	e_2	e_1+e_2	$2e_2$

$$\overset{a}{\underset{\Uparrow}{}}\ \overset{b}{\underset{\Uparrow}{}}\quad A+B:\ a+b$$

Theorem. In $\mathbb{R}^n$:

$$\operatorname{cvx}(v_1,\dots,v_k)+\operatorname{cvx}(w_1,\dots,w_m) = \operatorname{cvx}\Big(\underbrace{v_i+w_j : \begin{matrix}1\le i\le k,\\ 1\le j\le m\end{matrix}}_{mk \text{ points}}\Big)$$

g) Yes

B.

Sketch of proof: Start with

$$\overset{a}{t_1v_1+\dots+t_kv_k} \overset{+}{+} \overset{b}{s_1w_1+\dots+s_mw_m},\qquad \begin{matrix} t_1+\dots+t_k=1\\ s_1+\dots+s_m=1\\ t_j, s_i\geq 0.\end{matrix}$$

$\Big\downarrow$ algebraic manipulation

$$\sum_{\substack{1\le i\le k\\ 1\le j\le m}} c_{ij}(v_i+w_j), \qquad \sum c_{ij}=1, \quad c_{ij}\geq 0.$$