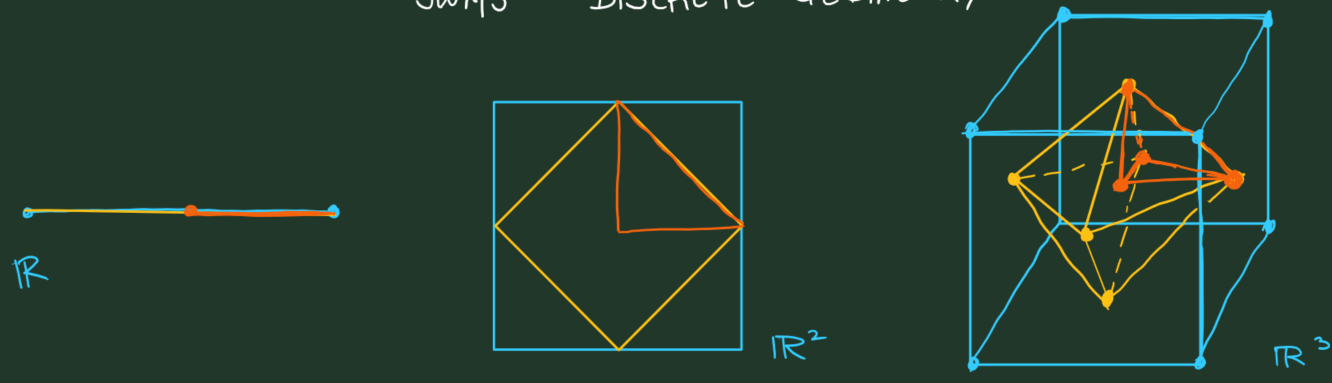
June 22, 2021

SWMS – DISCRETE GEOMETRY



LAST TIME: A convex polytope in $\mathbb{R}^n$ is the convex hull of a finite collection of points in $\mathbb{R}^n$.

Examples 1. The standard (hyper) cube in $\mathbb{R}^n$: $\underbrace{[-1,1] \times \cdots \times [-1,1]}_{n \text{ times}}$

= convex hull of all those points in $\mathbb{R}^n$ whose components/coordinates are $+1$ or -1.

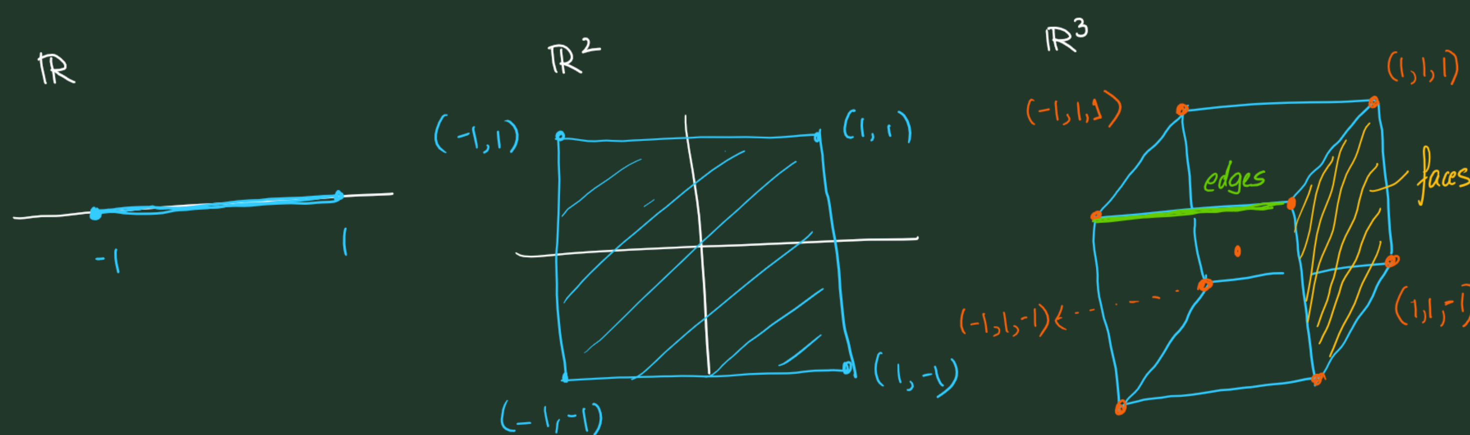
Q. How many such points are there in $\mathbb{R}^n$?

A. $(\underbrace{2 \times 2 \times 2 \cdots \times 2}_{n \text{ spaces}})$ where 2 = # of choices

$\therefore$ total no. of choices is 2^n!

$$= \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \max\{|x_1|, |x_2|, \dots, |x_n|\} \le 1 \}.$$

$$= \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \begin{array}{c} -1 \le x_1 \le 1 \\ \vdots \\ -1 \le x_n \le 1 \end{array} \right\}.$$



V: # vertices = 8

F: # faces = 6

E: # edges = 12

2. The crosspolytope in $\mathbb{R}^n$.

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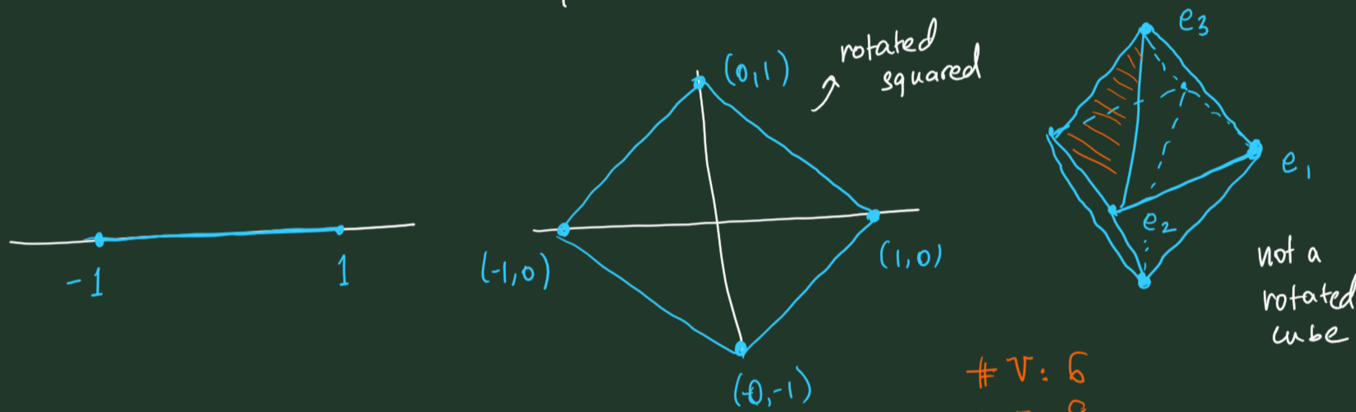
Let $e_j = (0, \dots, 1, \dots 0)$. $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

j^{th} position

The crosspolytope in $\mathbb{R}^n$ $= \operatorname{cvx}\{\pm e_1, \dots, \pm e_n\}$ 2n such vectors.

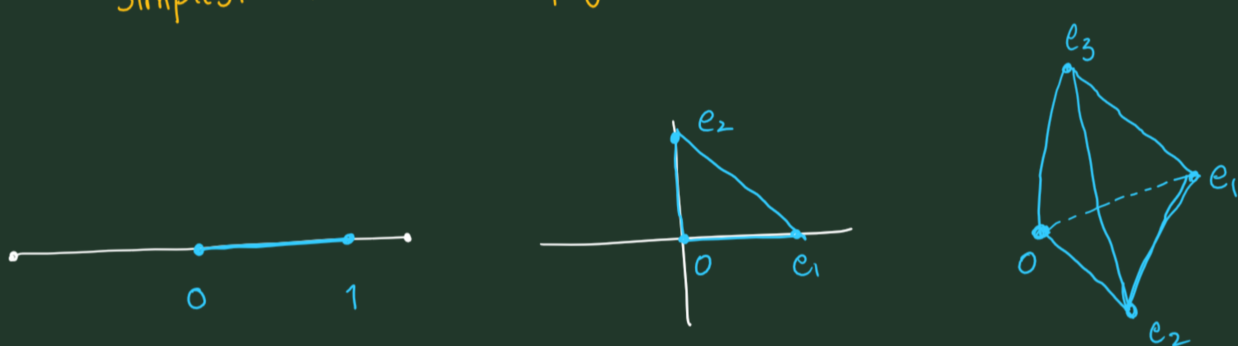
$= \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_1| + \dots + |x_n| \leq 1\}$.



V: 6
F: 8
E: 12

3. The standard simplex in $\mathbb{R}^n$ $= \operatorname{cvx}\{(0, \dots, 0), e_1, \dots, e_n\}$.

"simplest n-dim convex polytope in $\mathbb{R}^n$."

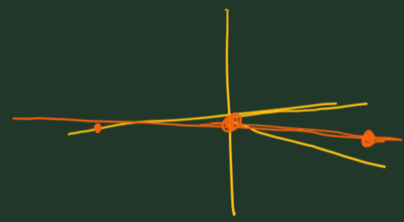


V: 4
F: 4
E: 6

In all the examples above in $\mathbb{R}^3$: $V - E + F = 2$.

FACT: any convex polytope in $\mathbb{R}^3$ satisfies this formula! Euler's formula.

Euler's formula.



Worksheet 2.

a) Null space: $\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$

Line passing through $(0,0,0)$ and $(-2,5,1)$.

$= \text{lin}\big((-2,5,1), \underline{(0,0,0)}\big) = \text{lin}\big((-2,5,1)\big)$.

Facts: $\text{lin}(v) = \begin{cases} \text{line passing through } v \text{ and } 0, & \text{if } v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$

b) Solution space of
$$x + 2z = 1$$
$$3x + y + z = 2.$$

$S = \{\underbrace{(1,-1,0) + t(-2,5,1)}_{} : t \in \mathbb{R}\}$ this is not a linear span! $0 \notin S$.

$= \overset{w_0}{(1,-1,0)} \overset{+}{+} \overset{\text{lin}(w_1)}{\text{lin}\big((-2,5,1)\big)} = \text{aff}(w_1 + w_0, w_0)$

$= \text{"aff}\big(\overset{(-1,4,1)}{\cancel{(-2,5,1)}}, (1,-1,0)\big)\text{"}$

$t_1 + t_2 = 1$
$t_1, t_2 \in \mathbb{R}$

$(x,y,z) = \underline{t(-2,5,1)} + (1-t)(1,-1,0)$,

$\{\}$
$t_1 = t$
$t_2 = 1 - t$

system of equations

$\text{aff}(w_1 + w_0, w_0) = \text{lin}(w_1) + w_0$

In worksheet 1: $\boxed{\text{aff}(v_1, v_2) = \text{lin}(\overset{w_1}{v_1 - v_2}) + \overset{w_0}{v_2}.}$

General: $\text{aff}(\underbrace{v_1}_{w_1 + w_k}, \dots, v_k) = \text{lin}(\underset{w_1}{v_1 - v_k}, \dots, \underset{w_{k-1}}{v_{k-1} - v_k}) + \underline{v_k}$

$\text{aff}(w_1 + w_k, \dots, w_{k-1} + w_k, w_k) = \text{lin}(w_1, \dots, w_{k-1}) + w_k$

(a) $Ax = 0$

(b) $Ax = b$

(a) $Ax = 0$

our case $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Null space: $\mathrm{lin}\left((-2,5,1)\right)$.

In general, null space of A is always a linear span.

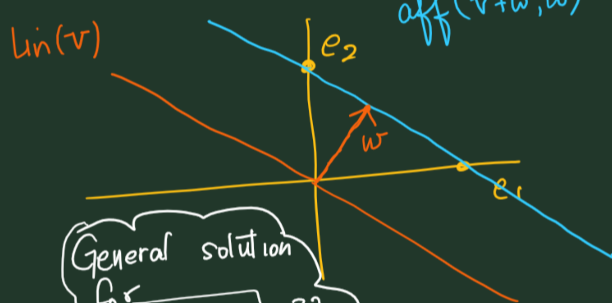
Solution space of (b) : particular solution + null space. $= (1,-1,0) + \mathrm{lin}\left((-2,5,1)\right)$

$$= \mathrm{aff}\left((-1,4,1),\ (1,-1,0)\right).$$

$$w_0\ \{\text{particular solution} + \mathrm{lin}\,(w_1, \ldots, w_m)$$

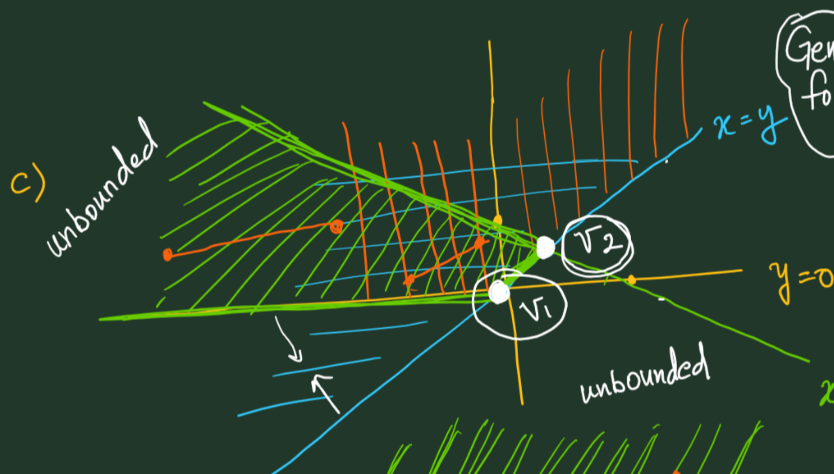
$$= \mathrm{aff}\,(w_1 + w_0, \ldots, w_m + w_0, w_0)$$

An affine space is always a linear span shifted by a vector.



General solution for $\boxed{Ax \leq b}$??

c)

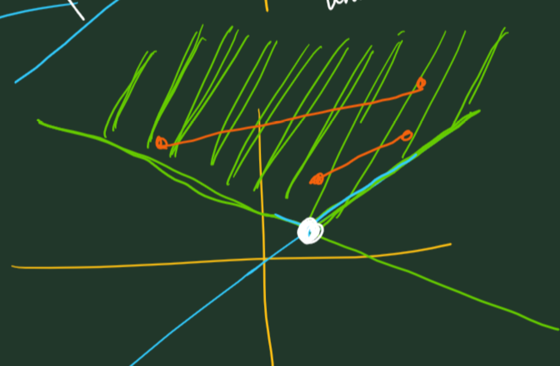


maybe convex hull

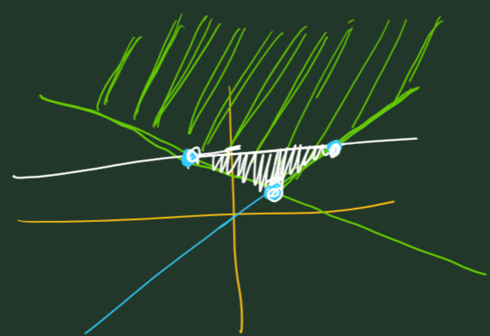
might not be a convex hull

No! These are not convex polytopes!

d)



d')



$x - y \le 0,\ y \ge 0,\ x + 2y \ge 1.$

add $y \le 1$

Based on c) and d), consider the following definitions:

$$\text{cone}(v_1, \dots, v_k) = \left\{ t_1 v_1 + \dots + t_k v_k \ : \ \begin{array}{l} t_1, \dots, t_k \in \mathbb{R} \\ t_1, \dots, t_k \ge 0 \end{array} \right\}$$

is the <u>polyhedral cone</u> of $v_1, \dots, v_k$ in $\mathbb{R}^n$.

c) & d) are polyhedral cones.

null space is trivial

c)

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x - y \\ -y \\ x + 2y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Downarrow$$

$$\left\{ \begin{array}{l} x - y \le 0 \\ -y \le 0 \\ x + 2y \le 1 \end{array} \right\}$$

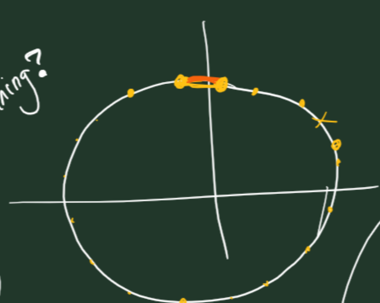
$Ax = b$ v/s $Ax \le b$

linear algebra — linear programming

solution space is an affine span. — solution space is a convex polytope / cone

e)

convex hull $(x^2 + y^2 = 1)$ meaning?



$x^2 + y^2 \le 1$ <u>unit disk</u>

[Guess: convex hull of <u>infinitely many</u> points]

Answer

not a convex hull of finitely many points.

$t_1 v_1 + \cdots t_k v_k + \cdots$?

Countable?

collection of all finite linear combinations?