SWMS 2021 – DISCRETE GEOMETRY

Let $v_1, \dots, v_k \in \mathbb{R}^n$ $(n = 2, 3)$.

1a) A linear combination of $v_1, \dots, v_k$ is any vector in $\mathbb{R}^n$ of the form

coefficients $t_1 v_1 + t_2 v_2 + \dots + t_k v_k$, where $t_1, \dots, t_k \in \mathbb{R}$.

b) An affine combination of $v_1, \dots, v_k$ is any vector in $\mathbb{R}^n$ of the form

$t_1 v_1 + \dots + t_k v_k$, where $t_1, \dots, t_k \in \mathbb{R}$ & $t_1 + \dots + t_k = 1$.

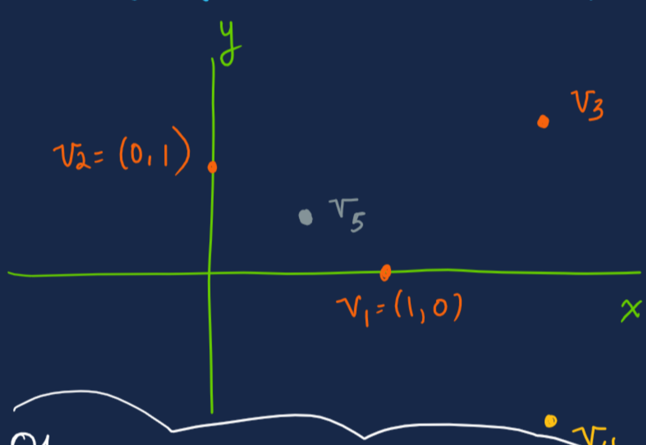
c) A convex combination of $v_1, \dots, v_k$ is any vector in $\mathbb{R}^n$ of the form

$t_1 v_1 + \dots + t_k v_k$, where $t_j \in \mathbb{R}$, $t_1 + \dots + t_k = 1$, & $t_j \geq 0, \forall j$.

Theorem

Every convex combination of $v_1, \dots, v_k$ is an affine combination of $v_1, \dots, v_k$.

Every affine combination of $v_1, \dots, v_k$ is a linear combination of $v_1, \dots, v_k$.



a) $v_3 = (2,1) = 2(1,0) + 1(0,1)$ is a linear, but not affine or cvx. combination of v_1 and v_2.

b) $v_4 = (2,-1) = 2(1,0) + (-1)(0,1)$

Note: $2 - 1 = 1$

v_4 is an affine (but not convex) combination of v_1 & v_2.

c) $v_5 = \left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(1,0) + \frac{1}{2}(0,1)$

is a convex combination of v_1 & v_2.

Q1.

Maybe, I can write

$V_4 = t_1(1,0) + t_2(0,1)$ for some $t_1 + t_2 = 1$, $t_1, t_2 \geq 0$? (form a basis.)

V_4 is not a cvx. combination of v_1 and v_2.

CANNOT HAPPEN! Because of the coordinates of v_1, v_2.

Q2. Is V_4 the convex combination of some other pair v_1, v_2?

$V_1 = V_4$

$V_2 = (0,0)$.

2a) The linear span / linear hull of $v_1, \dots, v_k$ is the set (in $\mathbb{R}^n$)

2a) The <u>linear span / linear hull</u> of $v_1, \dots, v_k$ is the set (in $\mathbb{R}^n$)

$$\text{lin}(v_1, \dots, v_k) = \left\{ \sum_{j=1}^{k} t_j v_j \ : \ t_j \in \mathbb{R} \right\}$$

all linear comb. of $v_1, \dots, v_k$.

b) The <u>affine hull</u> of $v_1, \dots, v_k$ is the set (in $\mathbb{R}^n$)

$$\text{aff}(v_1, \dots, v_k) = \left\{ \sum_{j=1}^{k} t_j v_j \ : \ t_j \in \mathbb{R} \ \forall j \ \& \ \sum_{j=1}^{k} t_j = 1 \right\}.$$

c) The <u>convex hull</u> of $v_1, \dots, v_k$ is the set (in $\mathbb{R}^n$)

$$\text{cvx}(v_1, \dots, v_k) = \left\{ \sum_{j=1}^{k} t_j v_j \ : \ t_j \in \mathbb{R}, \ t_j \geq 0, \ \sum_{j=1}^{k} t_j = 1 \right\}.$$

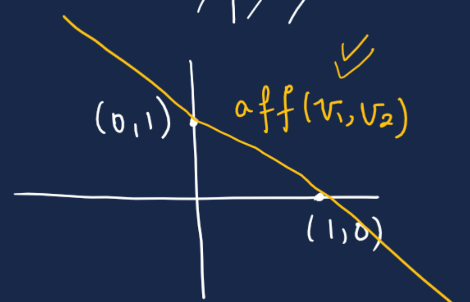
WORKSHEET 1

1a) $\text{lin}(v_1, v_2) = \{ t_1 v_1 + t_2 v_2 \ ; \ t_1, t_2 \in \mathbb{R} \}$ $\boxed{\text{Answer}}$

any $(x, y) \in x v_1 + y v_2$, where $x, y \in \mathbb{R}$.

b) $t_1 (1,0) + t_2 (0,1)$, $t_1 + t_2 = 1$

$= (t_1, 1 - t_1)$, $t_1 \in \mathbb{R}$.



all points here satisfy $x + y = 1$ and conversely any point on $x+y=1$ is of the form $(x, 1-x)$, $x \in \mathbb{R}$.

TRICK! any aff comb of v_1, v_2 : $t_1 v_1 + t_2 v_2$ and $t_1 + t_2 = 1$, $t_1, t_2 \in \mathbb{R}$

no freedom

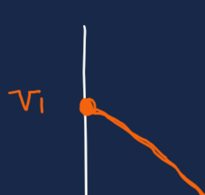
$t_1 = t \Rightarrow t_2 = 1 - t$

$\| $

$t v_1 + (1-t) v_2$, $t \in \mathbb{R}$.

1 degree of freedom

c) $\text{cvx}(v_1, v_2)$



Answer:

$\{ (x, y) : x, y \geq 0 \}$.

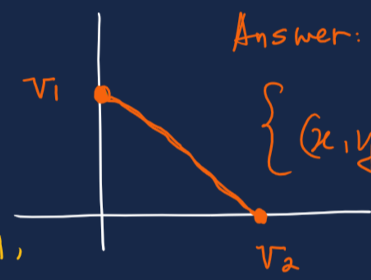
TRICK! Any convex combination :

c) $\mathrm{cvx}(v_1, v_2)$

1 degree of freedom

Answer:

$$\left\{ (x,y) : \begin{array}{l} x,y \geq 0 \\ x+y=1 \end{array} \right\}.$$



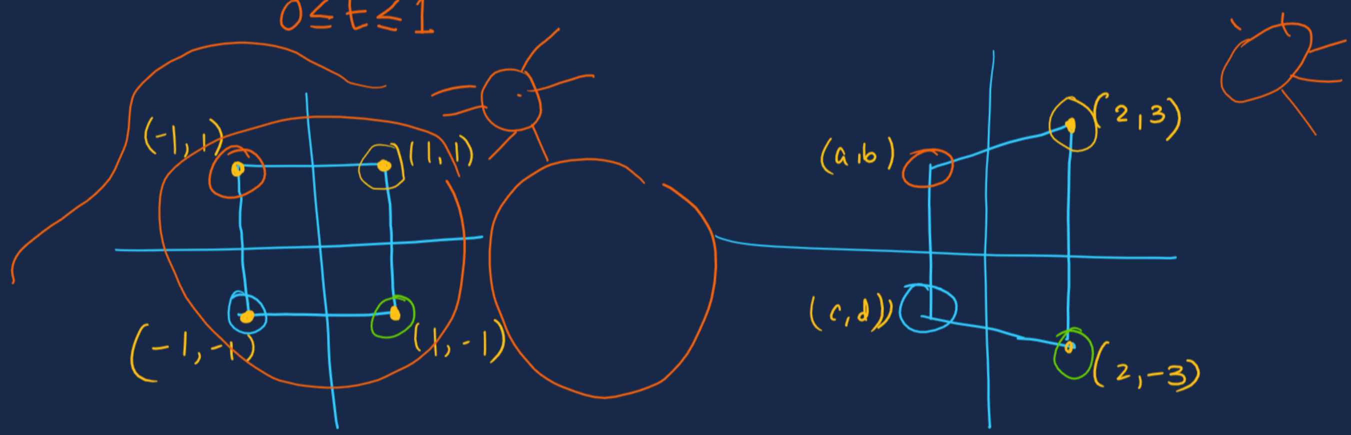
TRICK! Any convex combination:

$$t_1 v_1 + t_2 v_2 \qquad t_1+t_2=1, \quad t_1, t_2 \geq 0$$

$$||$$

$$t = t_1, \quad t_2 = 1-t$$

$$\underbrace{t v_1 + (1-t) v_2}_{0 \leq t \leq 1}, \qquad \begin{array}{l} 0 \leq t \\ 0 \leq 1-t \Rightarrow \underline{t \leq 1}. \end{array}$$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$